LINES OF MEAN CURVATURE ON SURFACES IMMERSED IN $\mathbb{R}^3$

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ABSTRACT. Associated to oriented surfaces immersed in $\mathbb{R}^3$, here are studied pairs of transversal foliations with singularities, defined on the *Elliptic* region, where the Gaussian curvature $\mathcal{K}$, given by the product of the principal curvatures k_1, k_2 of the immersion, is positive. The *leaves* of the foliations are the *lines of $\mathcal{M}$-mean curvature,* along which the normal curvature of the immersion is given by a function $\mathcal{M} = \mathcal{M}(k_1, k_2) \in [k_1, k_2]$, called a $\mathcal{M}$- *mean curvature*, whose properties extend and unify those of the *arithmetic* $\mathcal{H} = (k_1+k_2)/2$, the *geometric* $\sqrt{\mathcal{K}}$ and *harmonic* $\mathcal{K}/\mathcal{H} = ((1/k_1 + 1/k_2)/2)^{-1}$ *classical mean curvatures.*

The *singularities* of the foliations are the *umbilic points* and *parabolic curves*, where $k_1 = k_2$ and $\mathcal{K} = 0$, respectively.

Here are determined the patterns of $\mathcal{M}$- *mean curvature lines* near the *umbilic points*, *parabolic curves* and $\mathcal{M}$-*mean curvature cycles* (the periodic leaves of the foliations), which are structurally stable under small perturbations of the immersion. The genericity of these patterns is also established.

These patterns provide the three essential local ingredients to establish sufficient conditions, likely to be also necessary, for $\mathcal{M}$-*Mean Curvature Structural Stability* of immersed surfaces. This constitutes a natural unification and complement for the results obtained previously by the authors for the *Arithmetic*, [11], *Asymptotic*, [10, 14], *Geometric*, [12] and *Harmonic*, [13], *classical* cases of *Mean Curvature Structural Stability.*

1. INTRODUCTION

The study of families of curves defined for immersed surfaces by their normal curvature properties has attracted the interest of generations of mathematicians, among whom are Euler, Monge, Dupin, Gauss, Cayley, Darboux, Gullstrand, Caratheodory and Hamburger, to mention only a few. See [19, 32] for references.

Also dealing with families of curves, there is *"The Qualitative Theory of Differential Equations"* initiated by Poincaré and consolidated with the

study of the *Structural Stability and Genericity* of differential equations in the plane and surfaces, made systematic from 1937 to 1962 due to the seminal works of Andronov Pontrjagin and Peixoto; see [1, 26]. The basic ideas of this Theory were extended an applied by Gutierrez, Garcia and Sotomayor to principal curvature lines [16] as well as to other differential equations of classical geometry: the asymptotic lines [10, 14], the arithmetic, geometric and harmonic mean curvature lines [11, 12, 13].

An overview of the ensemble of the recent works cited above reveals that they share a common ground. In fact, there is a neat analogy in purpose, problems and methods of analysis. The goal of this paper is to inquire more deeply on their common features, possible mathematical discrepancies and limitations of the methods used so far.

In principle any expression such as $\mathcal{M} = \mathcal{M}(k_1, k_2) \in [k_1, k_2]$, involving the principal curvatures, could be rightly called a *"mean curvature"*. The solutions of the differential equation: $k_n([du, dv]) = \mathcal{M}$, would be called the *lines of $\mathcal{M}$-mean curvature*. Here, $k_n([du, dv]) = II(du, dv)/I(du, dv)$ is the normal curvature in the direction $([du, dv])$, as the quotient of the *second* and *first fundamental forms* of an immersed surface.

The situations that appear in the works quoted above correspond to the Principal Curvatures: $\mathcal{M} = k_1$ or $\mathcal{M} = k_2$ as well as to the Arithmetic, Geometric and Harmonic Mean Curvatures: $\mathcal{M} = \mathcal{H} = (k_1 + k_2)/2$, $\mathcal{M} = \mathcal{K}^{1/2}$, with $\mathcal{K} = k_1 k_2$, and $\mathcal{M} = \frac{\mathcal{K}}{\mathcal{H}}$. The asymptotic lines correspond to $\mathcal{M} = 0$ and are supported by the *hyperbolic* region of the immersion, where $\mathcal{K} < 0$. To these five functions we will refer to as the *"classical" mean curvature functions.* In the work of the authors the corresponding five quadratic differential equations and respective integral foliations with singularities have been unified in terms of their properties of structural stability under small perturbations of the immersed surface which supports them. A related unification, in terms of the notion of $T-$ *systems*, focusing the local form of the equations away from singularities, was proposed by Ogura in 1916; see [24].

In this work we extend to a general mean curvature function $\mathcal{M}$, with mild regularity assumptions, the essential results obtained for the "classical" mean curvature functions: principal lines [16], asymptotic lines [10, 14], arithmetic, geometric and harmonic mean curvature lines [11, 12, 13]. This generalization also includes interesting cases of *mean curvature functions* which seem to have been overlooked previously in Geometry.

To make precise the requirements imposed on a function $\mathcal{M} = \mathcal{M}(k_1, k_2)$ to be called a *mean curvature function*, it is appropriate to write its expression as $\mathcal{M} = m(H, K)$, in terms of a function m of the H, K variables, which when replaced by the *elementary symmetric functions* of (k_1, k_2): $H = \mathcal{H}(k_1, k_2)$ and $K = \mathcal{K}(k_1, k_2)$ give back $\mathcal{M}(k_1, k_2)$.

Definition 1. A function $\mathcal{M} = m(H, K)$ is called a mean curvature function provided the following holds:

1) It satisfies $(m - H)^2 \leq H^2 - K$ on the region $H^2 \geq K \geq 0$ (mean function condition).
2) It is continuous on the region $H^2 \geq K \geq 0$ and analytic on $K > 0$ (basic regularity condition).
3) $m(tH, t^2 K) = tm(H, K), \ t \geq 0$, (weighted homogeneity condition).

Remark 1. *In terms of (k_1, k_2), definition 1 amounts to the following:*

1) *$\mathcal{M} = \mathcal{M}(k_1, k_2) \in [k_1, k_2]$, is symmetric i.e. $\mathcal{M}(k_1, k_2) = \mathcal{M}(k_2, k_1)$,*
2) *it is continuous everywhere and analytic on $k_1 k_2 > 0$ and*
3) *it is homogeneous i.e. $\mathcal{M}(tk_1, tk_2) = t\mathcal{M}(k_1, k_2)$.*

Notice that $m(H, H^2) = H$, the diagonal condition, also expressed by $\mathcal{M}(k, k) = k$, follows directly from the mean function condition, labeled 1). One can also pass from $\mathcal{M}(k_1, k_2)$ to $m(H, K)$ by the transformation $k_1 = H - \sqrt{H^2 - K}$, $k_2 = H + \sqrt{H^2 - K}$

For more developments on the subject of *Means*, carried out from the perspective of Arithmetic and Analysis, the reader is addressed to Borwein and Borwein [4], Hardy et al. [20] and Mitrinovic [22]. Definition 1 is adapted for our needs from [4] , Chapter 8. Additional requirements will be made to it later to deal with differential geometric problems.

Definition 1 includes the classical symmetric means and also their most important generalizations, of which we review two examples below.

Example 1. The *Holder mean of order r*:

$$\mathcal{H}_r(k_1, k_2) = (\frac{k_1{}^r + k_2{}^r}{2})^{1/r}, \ r \neq 0; \ \ \mathcal{H}_0(k_1, k_2) = \sqrt{k_1 k_2}$$

generalizes the classical *arithmetic, geometric and harmonic mean curvatures*, given respectively by $r = 1, 0, -1$. The continuity at $r = 0$ and the limits $\lim_{r \to \pm\infty} \mathcal{H}_r = H \pm \sqrt{H^2 - K}$ are well known; see [20, 22].

Taking r as a parameter, this defines a natural transition between the classical means and consequently between their associated differential equations and foliations with singularities.

Example 2. The classical *AG mean* of Gauss and Legendre is defined by

$$\mathcal{AG}(k_1, k_2) = I(1, 1)/I(k_1, k_2); \ I(k_1, k_2) = \int_0^\infty \frac{dt}{(t^2 + k_1{}^2)^{1/2}(t^2 + k_2{}^2)^{1/2}}.$$

In Borwein and Borwein [4] can be found an enlightening study of this mean as well as a general treatment of the basic properties of Mean Functions. See also Weisstein [34] for other references on *means*, including recent, non symmetric, generalizations of the *AG mean*.

For any mean curvature function $\mathcal{M}$, as in definition 1, are defined two transversal foliations whose leaves, called the *lines of $\mathcal{M}$-mean curvature*, are the solutions of the quadratic differential equation $k_n([du, dv]) = \mathcal{M}$. These

foliations, called here the $\mathcal{M}$-*mean curvature foliations*, are well defined and regular only on the non-umbilic part of the elliptic region of the immersion, where the Gaussian Curvature is positive. The set where the Gaussian Curvature vanishes, the parabolic set, is generically a regular curve which is the border of the elliptic region. The umbilic points are those at which the principal curvatures coincide, generically are isolated and disjoint from the parabolic curve.

The transversal foliations, are assembled with the umbilic and parabolic points to define the $\mathcal{M}$-*mean curvature configuration* of an immersed surface. See section 2 for precise definitions.

This paper establishes sufficient conditions, likely to be also necessary, for the structural stability of $\mathcal{M}$-*mean curvature configurations*, under small perturbations of the immersion. See sections 2 and 6 for precise statements.

Three local ingredients are essential to express these sufficient conditions: the *umbilic points*, endowed with their $\mathcal{M}$-*mean curvature separatrix structure*, the $\mathcal{M}$-*mean curvature cycles*, with the calculation of the derivative of the Poincaré return map, through which is expressed the hyperbolicity condition and the *parabolic curve*, together with the *parabolic tangential singularities* and associated *separatrix structure*.

The conclusions of this work, on the *elliptic region*, are complementary to results valid independently for *asymptotic foliations* on the *hyperbolic region* (on which the Gaussian curvature is negative), for which the *separatrix structure* near the parabolic curve and the asymptotic structural stability has been studied in [10, 14].

This paper is organized as follows:

Section 2 is devoted to the general study of the differential equations and general properties of $\mathcal{M}$-Mean Curvature Lines. Here are given the precise definitions of the $\mathcal{M}$-Mean Curvature Configuration and of the two transversal $\mathcal{M}$-Mean Curvature Foliations with singularities into which it splits. The definition of $\mathcal{M}$-Mean Curvature Structural Stability focusing on the preservation of the qualitative properties of the foliations and the configuration under small perturbations of the immersion, will be given at the end of this section.

In Section 3 the equation of lines of $\mathcal{M}$-mean curvature is written in a Monge chart. The condition for the $\mathcal{M}$-mean curvature structural stability at umbilic points is explicitly stated in terms of the coefficients of the third order jet of the function which represents the immersion in a Monge chart. The local $\mathcal{M}$-mean curvature separatrix configurations at stable umbilics is established for C^4 immersions. The patterns resemble those established for the three Darbouxian umbilic points in the stable arithmetic mean curvature configurations [7, 16].

In Theorem 3 it is proved that this is due to the properties of mean curvature functions (definition 1 and remark 1). This clarifies why they

appear also in the geometric and harmonic mean curvature configurations studied previously in [12, 13].

In Section 4 is calculated the derivative of first return Poincaré map along a $\mathcal{M}$-mean curvature cycle. It consists of an integral expression involving $\mathcal{M}$ and other natural curvature functions along the cycle. Under an additional regularity condition on $\mathcal{M}$ (or m), denominated *positive regularity* in definition 2, it is shown how to deform an immersion so that a non hyperbolic $\mathcal{M}$-mean curvature cycle becomes hyperbolic.

In Section 5 are studied the foliations by lines of $\mathcal{M}$- mean curvature near the parabolic set of an immersion, which typically is a regular curve. Here it is also necessary to impose additional regularity conditions on the function $\mathcal{M}$. Two cases are considered in detail, denominated *1-regular* and *1/2-regular*. See definitions 3 and 4. In the 1-regular (resp. 1/2-regular) case, three (resp. only two) singular tangential patterns exist generically: the *folded node*, the *folded saddle* and the *folded focus* (resp. only the *folded node* and the *folded saddle*). The results of this section extend those obtained for the harmonic, as in [13], (resp. geometric, as in [12],) mean curvature.

In Section 6 the results presented in Sections 3, 4 and 5 are put together to provide sufficient conditions for $\mathcal{M}$-Mean Curvature Structural Stability.

The density of these conditions is established in section 7. The delicate point here is the elimination of non-trivial recurrent $\mathcal{M}$-mean curvature lines by means of small perturbations of the immersions. The main steps for the somewhat technical proof of this part are explained in detail here under suitable hypotheses.

Section 8 presents a short overview of the achievements of this paper and points out to some possible lines for future research.

For a discussion on historic grounds of the prominence of the classical means in Arithmetic, Geometry and Analysis and of the needs for their generalization, the reader is addressed to the book by Borwein and Borwein, [4]. See also the essay by Wassell, [33].

2. Differential Equations for $\mathcal{M}$- Mean Curvature Lines

Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be a C^r, $r \geq 4$, immersion of an oriented smooth surface $\mathbb{V}^2$ into $\mathbb{R}^3$. This means that $D\alpha$ is injective at every point in $\mathbb{V}^2$.

The space $\mathbb{R}^3$ is oriented by a once for all fixed orientation and endowed with the Euclidean inner product $<,>$.

Let N be the positive unit vector field normal to α. This means that for any positive chart (u,v) of $\mathbb{V}^2$, $\{\alpha_u, \alpha_v, N\}$ is a positive frame in $\mathbb{R}^3$.

In such chart (u,v), the *first fundamental form* of an immersion α is given by:
$$I_\alpha = < D\alpha, D\alpha >= Edu^2 + 2Fdudv + Gdv^2,$$
with $E = < \alpha_u, \alpha_u >$, $F = < \alpha_u, \alpha_v >$, $G = < \alpha_v, \alpha_v >$

The *second fundamental form* is given by:
$$II_\alpha = < N, D^2\alpha >= edu^2 + 2fdudv + gdv^2,$$

with $e = <N, \alpha_{uu}> = -<N_u, \alpha_u>$, $f = <N, \alpha_{uv}> = -<N_u, \alpha_v>$, $g = <N, \alpha_{vv}> = -<N_v, \alpha_v>$.

The normal curvature at a point p in a tangent direction $t = [du : dv]$ is given by:

$$k_n = k_n(p) = \frac{II_\alpha(t,t)}{I_\alpha(t,t)}.$$

Given a mean curvature function $\mathcal{M}$ as in definition 1 and remark 1, the lines of $\mathcal{M}$-mean curvature of α are regular curves on $\mathbb{V}^2$ along which the normal curvature is equal to $\mathcal{M}$. That is, $k_n = \mathcal{M}(k_1, k_2) = m(\mathcal{H}, \mathcal{K})$, where $\mathcal{K} = \mathcal{K}_\alpha$ and $\mathcal{H} = \mathcal{H}_\alpha$ are the Gaussian and Arithmetic Mean curvatures of α. See definition 1.

Therefore the pertinent differential equation for these lines is given by:

$$\frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \mathcal{M}(k_1, k_2) = m(\mathcal{H}, \mathcal{K})$$

where $\mathcal{H} = \frac{Eg + eG - 2fF}{2(EG - F^2)}$ and $\mathcal{K} = \frac{eg - f^2}{EG - F^2}$ or equivalently, according to remark 1, expressing in $\mathcal{M}(k_1, k_2)$ the principal curvatures in terms of $(\mathcal{H}, \mathcal{K})$.

Or equivalently by

$$(g - \mathcal{M}G)dv^2 + 2(f - \mathcal{M}F)dudv + (e - \mathcal{M}E)du^2 = 0. \qquad (1)$$

This equation is well defined only on the closure of the *Elliptic region*, $\mathbb{E}\mathbb{V}^2_\alpha$, of α, where $\mathcal{K} > 0$. It is bivalued and C^{r-2}, $r \geq 4$, smooth on the complement of the umbilic, $\mathcal{U}_\alpha$, and parabolic, $\mathcal{P}_\alpha$, sets of the immersion α. In fact, on $\mathcal{U}_\alpha$, where the principal curvatures coincide, i.e where $\mathcal{H}^2 - \mathcal{K} = 0$, the equation vanishes identically; on $\mathcal{P}_\alpha$, it is univalued when $k_2 = \mathcal{M}$ or when $\mathcal{M} = k_1 = 0$.

The lines of $\mathcal{M}$-mean curvature of immersions will be organized into the *$\mathcal{M}$-mean curvature configuration,* as follows:

Through every point $p \in \mathbb{E}\mathbb{V}^2{}_\alpha \setminus (\mathcal{U}_\alpha \cup \mathcal{P}_\alpha)$, pass two $\mathcal{M}$-mean curvature lines of α. Under the orientability hypothesis imposed on $\mathbb{V}^2$, the $\mathcal{M}$-mean curvature lines define two foliations: $\mathbb{H}_{\alpha,1}$, called the $\mathcal{M}$-*minimal mean curvature foliation,* along which the *geodesic torsion* is negative (i.e $\tau_g = -\sqrt{(k_2 - \mathcal{M})(\mathcal{M} - k_1)}$) and $\mathbb{H}_{\alpha,2}$, called the $\mathcal{M}$-*maximal mean curvature foliations,* along which the geodesic torsion is positive (i.e $\tau_g = \sqrt{(k_2 - \mathcal{M})(\mathcal{M} - k_1)}$)

By comparison with the arithmetic mean curvature directions, making angle $\pi/4$ with the minimal principal directions, the $\mathcal{M}$ directions are located between them and the principal ones, making an angle θ_m such that $\tan \theta_m = \pm\sqrt{\frac{\mathcal{M} - k_1}{k_2 - \mathcal{M}}}$, as follows from Euler's Formula: $k_n(\theta) = k_1 \cos^2(\theta) + k_2 \sin^2(\theta)$,

which leads directly to $\cos\theta_m = \pm\sqrt{\frac{\mathcal{M}-k_1}{k_2-k_1}}$ and $\sin\theta_m = \pm\sqrt{\frac{k_2-\mathcal{M}}{k_2-k_1}}$. The particular expression for the geodesic torsion given above results from the formula $\tau_g = (k_2 - k_1)\sin\theta\cos\theta$ [32]. See also lemma 1 in Section 4 below.

The quadruple
$$\mathbb{H}_\alpha^\mathcal{M} = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{H}_{\alpha,1}^\mathcal{M}, \mathbb{H}_{\alpha,2}^\mathcal{M}\}$$
is called the $\mathcal{M}$-*mean curvature configuration* of α.

It splits into two foliations with singularities:
$$\mathbb{G}_{\alpha,i}^\mathcal{M} = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{H}_{\alpha,i}^\mathcal{M}\}, \; i = 1, 2.$$

Let $\mathbb{V}^2$ be also compact. Denote by $\mathcal{I}^{r,s}(\mathbb{V}^2)$ the space of C^r immersions of $\mathbb{V}^2$ into the Euclidean space $\mathbb{R}^3$, endowed with the C^s topology.

An immersion α is said C^s-$\mathcal{M}$-*local mean curvature structurally stable at a compact set* $C \subset \mathbb{V}^2$ if for any sequence of immersions α_n converging to α in $\mathcal{I}^{r,s}(\mathbb{V}^2)$ there is a neighborhood V_C of C, sequence of compact subsets C_n and a sequence of homeomorphisms mapping C to C_n, converging to the identity of $\mathbb{V}^2$ such that on V_C it maps umbilic and parabolic points and arcs of the $\mathcal{M}$-mean curvature foliations $\mathbb{H}_{\alpha,i}$ to those of $\mathbb{H}_{\alpha_n,i}$ for $i = 1$, 2.

An immersion α is said to be C^s-$\mathcal{M}$-*mean curvature structurally stable* if the compact C above is the closure of $\mathbb{E}\mathbb{V}^2{}_\alpha$.

Analogously, α is said to be *i- C^s-$\mathcal{M}$-mean curvature structurally stable* if only the preservation of elements of *i-th, i=1,2* foliation with singularities is required.

A general study of the structural stability of quadratic differential equations (not necessarily derived from normal curvature properties) has been carried out by Guíñez [15]. See also the work of Bruce and Fidal [5] , Bruce and Tari [6] and Davydov [8] for the analysis of umbilic points for general quadratic and also implicit differential equations.

For a study of the topology of foliations with non-orientable singularities on two dimensional manifolds, see the works of Rosenberg and Levitt [29, 21]. In these works the leaves are not defined by normal curvature properties.

3. $\mathcal{M}$-MEAN CURVATURE LINES NEAR UMBILIC POINTS

Let 0 be an umbilic point of a C^r, $r \geq 4$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where

$$z(x, y) = \frac{k}{2}(x^2 + y^2) + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4) \tag{2}$$

This reduced form is obtained by means of a rotation of the x, y-axes. See [16, 18].

Proposition 1. *Assume the notation established in equation 2. Suppose that the transversality condition* $T_m : kb(b - a) \neq 0$ *holds and consider the following situations:*

$M_1)$ $\Delta_m > 0$

M_2) $\Delta_m < 0$ *and* $\dfrac{a}{b} > 1$

M_3) $\dfrac{a}{b} < 1$.

Here $\Delta_m = 4c^2(2a - b)^2 - [3c^2 + (a - 5b)^2][3(a - 5b)(a - b) + c^2]$.

Then for every mean curvature function $\mathcal{M}$, *the foliations* $\{\mathbb{H}_1^{\mathcal{M}}, \mathbb{H}_2^{\mathcal{M}}\}$ *have in a neighborhood of* 0, *one hyperbolic sector in the* M_1 *case, one parabolic and one hyperbolic sector in* M_2 *case and three hyperbolic sectors in the case* M_3. *These points are called Darbouxian umbilics, see Figure 1.*

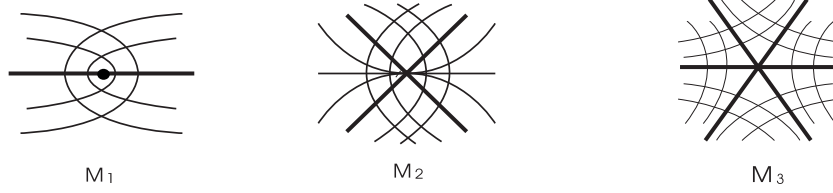*The separatrices of these singularities are called umbilic separatrices.*

M_1 M_2 M_3

FIGURE 1. $\mathcal{M}$-mean curvature lines near the umbilic points M_i and their separatrices

Proof. Near 0, the functions $\mathcal{K}$ and $\mathcal{H}$ have the following Taylor expansions.

$$\mathcal{K} = k^2 + (a + b)kx + cky + O_1(2), \quad \mathcal{H} = k + \frac{1}{2}(a + b)x + \frac{1}{2}cy + O_2(2).$$

Assume that $\mathcal{M}$ is a mean curvature function, as defined in 1 and therefore from $(m - H)^2 \le H^2 - K$ follows that $j^1 \mathcal{M}_{(H,H^2)}(0,0) = j^1 \mathcal{H}_{(H,H^2)}(0,0)$.

The differential equation of the $\mathcal{M}$-mean curvature lines

$$[g - \mathcal{M}G]dx^2 + 2[f - \mathcal{M}F]dxdy + [e - \mathcal{M}E]dy^2 = 0 \tag{3}$$

is given by:

$$\begin{aligned} &[(b - a)x + cy + R_1(x,y)]dy^2 + [4by + R_2(x,y)]dxdy \\ &- [(b - a)x + cy + R_3(x,y)]dx^2 = 0 \end{aligned} \tag{4}$$

where R_i, $i = 1, 2, 3$, represent functions of order $O((x^2 + y^2))$.

Thus, at the level of first jet, the differential equation 4 is the same as that of the arithmetic mean curvature lines given by

$$[g - \mathcal{H}G]dv^2 + 2[f - \mathcal{H}F]dudv + [e - \mathcal{H}E]du^2 = 0.$$

The conditions on Δ_m coincide with those on Δ_H, established to characterize the arithmetic mean curvature Darbouxian umbilics studied in detail in [11]. Thus reducing the analysis of the umbilic points to that of the hyperbolicity of saddles and nodes of plane vector fields, whose phase portraits are determined only by the first jets of the vector field at the singularities, which are calculated only in terms of the first jet of the equation 4 at the umbilic point. $\qquad\square$

Theorem 3. An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 4$, is $C^3 - \mathcal{M}$-local mean curvature structurally stable at $\mathcal{U}_\alpha$ if and only if every $p \in \mathcal{U}_\alpha$ is one of the types M_i, $i = 1, 2, 3$ of proposition 1.

Proof. Clearly proposition 1 shows that the condition M_i, $i = 1, 2, 3$ together with $T_m : kb(b - a) \neq 0$ imply the $C^3- \mathcal{M}$-local mean curvature structural stability. This involves the construction of the homeomorphism (by means of canonical regions), mapping simultaneously minimal and maximal $\mathcal{M}$-mean curvature lines around the umbilic points of α onto those of a C^4 slightly perturbed immersion.

We will discuss the necessity of the condition $T_m : k(b - a)b \neq 0$ and of the conditions M_i, $i = 1, 2, 3$. The first one follows from its identification with a transversality condition that guarantees the persistent isolation of the umbilic points of α and its separation from the parabolic set, as well as the persistent regularity of the Lie-Cartan surface $\mathcal{G}$, obtained from the projectivization of the equation 3. Failure of T_m condition has the following implications:

a) $b(b - a) = 0$; in this case the elimination or splitting of the umbilic point can be achieved by small perturbations.
b) $k = 0$ and $b(b - a) \neq 0$; in this case a small perturbation separates the umbilic point from the parabolic set.

The necessity of condition M_i follows from its dynamic identification with the hyperbolicity of the equilibria along the projective line of the vector field obtained lifting equation (3) to the surface $\mathcal{G}$. Failure of this condition would make possible to change the number of $\mathcal{M}$-mean curvature separatrices at the umbilic point by means a small perturbation of the immersion. $\square$

4. Periodic $\mathcal{M}$-Mean Curvature Lines

Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of a compact and oriented surface and consider the foliations $\mathbb{H}_{\alpha,i}^{\mathcal{M}}$, $i = 1, 2$, given by the $\mathcal{M}$-*mean curvature lines*.

In terms of $\mathcal{M}$ and other natural geometric invariants of the immersion, here is established an integral expression for the first derivative of the return map of a periodic $\mathcal{M}$-mean curvature line, called also a $\mathcal{M}$-*mean curvature cycle*. Recall that the return map associated to a cycle is a local diffeomorphism with a fixed point, defined on a cross section normal to the cycle by following the integral curves through this section until they meet again the section. This map is called holonomy in Foliation Theory and Poincaré Map in Dynamical Systems, [23].

A $\mathcal{M}$-mean curvature cycle is called *hyperbolic* if the first derivative of the return map at the fixed point is different from one.

The $\mathcal{M}$-mean curvature foliations $\mathbb{H}_{\alpha,i}$ have no cycles such that the return map reverses the orientation. Initially, the integral expression for the derivative of the return map is obtained in class C^6; see Lemma 2 and Proposition 2.

The characterization of hyperbolicity of $\mathcal{M}$-mean curvature cycles in terms of local structural stability is given in Theorem 4 of this section.

Lemma 1. *Let $c : I \to \mathbb{V}^2$ be a $\mathcal{M}$- mean curvature line parametrized by arc length. Then the Darboux frame is given by:*

$$T' = k_g N \wedge T + \mathcal{M}N$$

$$(N \wedge T)' = -k_g T + \tau_g N$$

$$N' = -\mathcal{M}T - \tau_g N \wedge T$$

where $\tau_g = \pm\sqrt{(\mathcal{M} - k_1)(k_2 - \mathcal{M})}$. The sign of τ_g is positive (resp. negative) if c is maximal (resp. minimal) $\mathcal{M}$-mean curvature line.

Proof. The normal curvature k_n of the curve c is by the definition the mean curvature function $\mathcal{M}$. ¿From the Euler equation $k_n = k_1 \cos^2 \theta + k_2 \sin^2 \theta = \mathcal{M}$, get $\tan \theta = \pm\sqrt{\frac{\mathcal{M} - k_1}{k_2 - \mathcal{M}}}$.

Therefore, by direct calculation, the geodesic torsion is given by $\tau_g = (k_2 - k_1) \sin \theta \cos \theta = \pm\sqrt{(\mathcal{M} - k_1)(k_2 - \mathcal{M})}$. $\qquad\square$

Remark 2. *The expression for the geodesic curvature k_g will not be needed explicitly in this work. However, it can be given in terms of the principal curvatures and their derivatives using a formula due to Liouville [32], pp. 130-131. In fact, in a principal chart (u, v) the geodesic curvatures of the coordinate curves are given by:*

$$k_g|_{v=v_0} = \frac{\frac{\partial k_1}{\partial v}}{k_1 - k_2}, \quad k_g|_{u=u_0} = \frac{\frac{\partial k_2}{\partial u}}{k_1 - k_2}.$$

Therefore, by Liouville formula, the geodesic curvature of a curve $c(s)$ parametrized by arc length and that makes an angle $\theta(s)$ with the principal direction $e_1 = \partial/\partial u$ is

$$k_g = \frac{d\theta}{ds} + k_g|_{v=v_0} \cos \theta + k_g|_{u=u_0} \sin \theta.$$

Lemma 2. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^r, $r \geq 6$, and c be a $\mathcal{M}$ - mean curvature cycle of α, parametrized by arc length and of length L. Then the expression,*

$$\alpha(s, v) = c(s) + v(N \wedge T)(s) + [(2\mathcal{H}(s) - \mathcal{M}(s))\frac{v^2}{2} + \frac{A(s)}{6}v^3 + v^3 B(s, v)]N(s)$$

where $B(s, 0) = 0$, defines a local chart (s, v) of class C^{r-5} in a neighborhood of c.

Proof. The curve c is of class C^{r-1} and the map $\alpha(s, v, w) = c(s) + v(N \wedge T)(s) + wN(s)$ is of class C^{r-2} and is a local diffeomorphism in a neighborhood of the axis s. In fact $[\alpha_s, \alpha_v, \alpha_w](s, 0, 0) = 1$. Therefore there is a function $W(s, v)$ of class C^{r-2} such that $\alpha(s, v, W(s, v))$ is a parametrization of a tubular neighborhood of $\alpha \circ c$. Now for each s, $W(s, v)$ is just a

parametrization of the curve of intersection between $\alpha(\mathbb{V}^2)$ and the normal plane generated by $\{(N \wedge T)(s), N(s)\}$. This curve of intersection is tangent to $(N \wedge T)(s)$ at $v = 0$ and notice that $k_n(N \wedge T)(s) = 2\mathcal{H}(s) - \mathcal{M}(s)$. Therefore,

$$
\begin{aligned}
\alpha(s, v, W(s, v)) =& c(s) + v(N \wedge T)(s) \\
&+ [(2\mathcal{H}(s) - \mathcal{M}(s))\frac{v^2}{2} + \frac{A(s)}{6}v^3 + v^3 B(s, v)]N(s),
\end{aligned}
\tag{5}
$$

where A is of class C^{r-5} and $B(s, 0) = 0$. $\qquad\square$

We now compute the coefficients of the first and second fundamental forms in the chart (s, v) constructed above, to be used in proposition 2.

$$
\begin{aligned}
N(s, v) =& \frac{\alpha_s \wedge \alpha_v}{|\alpha_s \wedge \alpha_v|} = [-\tau_g(s)v + O(2)]T(s) \\
&- [(2\mathcal{H}(s) - \mathcal{M}(s))v + O(2)](N \wedge T)(s) + [1 + O(2)]N(s).
\end{aligned}
$$

Therefore it follows that $E = <\alpha_s, \alpha_s>$, $F = <\alpha_s, \alpha_v>$, $G = <\alpha_v, \alpha_v>$, $e = <N, \alpha_{ss}>$, $f = <N, \alpha_{sv}>$ and $g = <N, \alpha_{vv}>$ are given by

$$
\begin{aligned}
E(s, v) &= 1 - 2k_g(s)v + h.o.t \\
F(s, v) &= 0 + 0.v + h.o.t \\
G(s, v) &= 1 + 0.v + h.o.t \\
e(s, v) &= \mathcal{M}(s) + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s)] + h.o.t \\
f(s, v) &= \tau_g(s) + \{[2\mathcal{H}(s) - \mathcal{M}(s)]' + k_g(s)\tau_g(s)\}v + h.o.t \\
g(s, v) &= 2\mathcal{H}(s) - \mathcal{M}(s) + A(s)v + h.o.t
\end{aligned}
\tag{6}
$$

Proposition 2. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^r, $r \geq 6$ and c be closed $\mathcal{M}$-mean curvature line c of α, parametrized by arc length s and of total length L. Then the derivative of the Poincaré map π_α associated to c is given by:*

$$
ln\pi_\alpha'(0) = \int_0^L \left[\frac{[\mathcal{M}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \mathcal{M}) \right] ds.
$$

Here $\tau_g = \pm\sqrt{(\mathcal{M} - k_1)(k_2 - \mathcal{M})}$.

Proof. The Poincaré map associated to c is the map $\pi_\alpha : \Sigma \to \Sigma$ defined in a transversal section to c such that $\pi_\alpha(p) = p$ for $p \in c \cap \Sigma$ and $\pi_\alpha(q)$ is the first return of the $\mathcal{M}$-mean curvature line through q to the section Σ, choosing a positive orientation for c. It is a local diffeomorphism and is defined, in the local chart (s, v) introduced in Lemma 2, by $\pi_\alpha : \{s = 0\} \to \{s = L\}$, $\pi_\alpha(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of the Cauchy problem

$$
(g - \mathcal{M})dv^2 + 2(f - \mathcal{M}F)dsdv + (e - \mathcal{M}E)ds^2 = 0, \quad v(0, v_0) = v_0.
$$

Direct calculation gives that the derivative of the Poincaré map satisfies the following linear differential equation:

$$\frac{d}{ds}\left(\frac{dv}{dv_0}\right) = -\frac{N_v}{M}\left(\frac{dv}{dv_0}\right) = -\frac{[e - \mathcal{M}E]_v}{2[f - \mathcal{M}F]}\left(\frac{dv}{dv_0}\right)$$

Therefore, using equation 6 it results that

$$\frac{[e - \mathcal{M}E]_v}{2[f - \mathcal{M}F]} = -\frac{\tau_g'}{2\tau_g} - \frac{[\mathcal{M}]_v}{2\tau_g} - \frac{k_g}{\tau_g}(\mathcal{H} - \mathcal{M}).$$

Integrating the equation above along an arc $[s_0, s_1]$ of $\mathcal{M}$-mean curvature line, it follows that:

$$\frac{dv}{dv_0}\Big|_{v_0 = 0} = \frac{(\tau_g(s_1))^{\frac{-1}{2}}}{(\tau_g(s_0))^{\frac{-1}{2}}} exp[\int_{s_0}^{s_1}\left[\frac{[\mathcal{M}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \mathcal{M})\right]ds. \qquad (7)$$

Applying 7 along the $\mathcal{M}$-mean curvature cycle of length L, obtain

$$\frac{dv}{dv_0}\Big|_{v_0 = 0} = exp[\int_0^L\left[\frac{[\mathcal{M}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \mathcal{M})\right]ds.$$

¿From the equation $\mathcal{K} = (eg - f^2)/(EG - F^2)$ evaluated at $v = 0$, it follows using the expressions in 6 that $\mathcal{K} = \mathcal{M}[2\mathcal{H} - \mathcal{M}] - \tau_g^2$. Developing this equation it follows that $\tau_g = \pm\sqrt{(\mathcal{M} - k_1)(k_2 - \mathcal{M})}$.
This ends the proof. □

Remark 3. *The study of the behavior of curvature lines near principal cycles was carried out in [16], [18] and [9]. In this last work was established the general integral pattern for the successive derivatives of the return map.*

For the next theorem it is necessary to assume the additional property of being *positive regular* for the function $\mathcal{M} = m(H, K)$.

Definition 2. A mean curvature function $\mathcal{M} = m(H, K)$, as in definition 1, is called *positive regular* if

$$\overline{\mathcal{M}} = \mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K > 0.$$

Proposition 3. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^r, $r \geq 6$, and c be a maximal $\mathcal{M}$ - mean curvature cycle for α, parametrized by arc length and of length L. Consider a chart (s, v) as in lemma 2 and consider the deformation*

$$\beta_\epsilon(s, v) = \beta(\epsilon, s, v) = \alpha(s, v) + \epsilon[\frac{A_1(s)}{6}v^3]\delta(v)N(s)$$

where $\delta = 1$ in neighborhood of $v = 0$, with small support and $A_1(s) = \tau_g(s) > 0$.
Then c is a $\mathcal{M}$-mean curvature cycle of β_ϵ for all ϵ small. Also, provided $\mathcal{M}$ is positive regular, that is in definition 2) $\overline{\mathcal{M}} > 0$, c is a hyperbolic $\mathcal{M}$-mean curvature cycle for β_ϵ, and $\epsilon \neq 0$ small.

Proof. In the chart (s, v), for the immersion β_ϵ, it is obtained that:

$$E_\epsilon(s, v) = 1 - 2k_g(s)v + h.o.t$$
$$F_\epsilon(s, v) = 0 + 0.v + h.o.t$$
$$G_\epsilon(s, v) = 1 + 0.v + h.o.t$$
$$e_\epsilon(s, v) = \mathcal{M}(s) + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s))] + h.o.t$$
$$f_\epsilon(s, v) = \tau_g(s) + [(2\mathcal{H}(s) - \mathcal{M}(s))' + k_g\tau_g]v + h.o.t$$
$$g_\epsilon(s, v) = 2\mathcal{H}(s) - \mathcal{M}(s) + v[A(s) + \epsilon A_1(s)] + h.o.t$$

In the expressions above $E_\epsilon = < \beta_s, \beta_s >$, $F_\epsilon = < \beta_s, \beta_v >$, $G_\epsilon = < \beta_v, \beta_v >$, $e_\epsilon = < \beta_{ss}, N >$, $f_\epsilon = < N, \beta_{sv} >$, $g_\epsilon = < N, \beta_{vv} >$, where $N = N_\epsilon = \beta_s \wedge \beta_v / \mid \beta_s \wedge \beta_v \mid$.

Let $\mathcal{M}_\epsilon = m(\mathcal{H}_\epsilon, \mathcal{K}_\epsilon)$. For all ϵ small it follows that:

$$(e_\epsilon - \mathcal{M}_\epsilon E_\epsilon)(s, 0, \epsilon) = 0$$
$$\mathcal{K}_{\epsilon v}(s, 0, \epsilon) = \epsilon \mathcal{M}_\epsilon A_1(s) + f_1(k_g, \tau_g, \mathcal{K}, \mathcal{H})(s)$$
$$\mathcal{H}_{\epsilon v}(s, 0, \epsilon) = \frac{1}{2}\epsilon A_1(s) + f_2(k_g, \tau_g, \mathcal{K}, \mathcal{H})(s)$$
$$\frac{d}{d\epsilon}[\mathcal{M}_\epsilon]_v|_{\epsilon=0} = \frac{1}{2}[\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K]]A_1(s).$$

Therefore c is a maximal $\mathcal{M}$-mean curvature cycle for all β_ϵ.

Assuming that $A_1(s) = 4\tau_g(s) > 0$, and also that $\mathcal{M}$ is *positive regular*, i.e. by definition 2,

$$\overline{\mathcal{M}} = \mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K > 0,$$

it results that

$$\frac{d}{d\epsilon}(ln\pi'(0))|_{\epsilon=0} = \int_0^L \frac{d}{d\epsilon}\left(\frac{(\mathcal{M}_\epsilon)_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H}_\epsilon - \mathcal{M}_\epsilon)\right) ds$$
$$= \int_0^L \tau_g\overline{\mathcal{M}}ds > 0.$$

$\square$

As a synthesis of propositions 2 and 3, the following theorem is obtained.

Theorem 4. Let $\mathcal{M}$ be a positive regular mean curvature function. An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 6$, is C^6-local $\mathcal{M}$-mean curvature structurally stable at a $\mathcal{M}$-mean curvature cycle c if only if,

$$\int_0^L \left[\frac{[\mathcal{M}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \mathcal{M})\right] ds \neq 0.$$

Proof. Using propositions 2 and 3, the local topological character of the foliation can be changed by small perturbation of the immersion, when the cycle is not hyperbolic. $\square$

5. $\mathcal{M}$-Mean Curvature Lines near the Parabolic Curve

In this section will be studied the behavior of the $\mathcal{M}$-mean curvature lines near the parabolic points of an immersion, assuming that the quadratic differential equation 1 is univalued there. This is done under two regularity conditions imposed in definitions 3 and 4. The motivation comes from the previous study of the classical harmonic and geometric mean curvature functions; see [13, 11].

Definition 3. A mean curvature function $\mathcal{M} = m(H, K)$ is called *1-regular* if either

    a) $m(H, 0) = 0$ and $(\partial m / \partial K)(H, 0) > 1/(2H) > 0$, or
    b) $m(H, 0) = 2H$ and $(\partial m / \partial K)(H, 0) < -1/(2H) < 0$.

Remark 4. *For mean curvature functions m, with $m(H, 0) = 0$, it always holds that $(\partial m / \partial K)(H, 0) \geq 1/(2H) > 0$. The $1-$ regular condition states that the inequality is strict. In fact,*

$$\frac{\partial m}{\partial K}(H, 0) = \lim_{K \to 0} \frac{m(H, K) - M(H, 0)}{K} \geq \lim_{K \to 0} \frac{H - \sqrt{H^2 - K}}{K} = \frac{1}{2H}.$$

Analogously for the case where $m(H, 0) = 2H$.

Definition 4. A mean curvature function $\mathcal{M} = m(H, K)$ is called *1/2-regular* if either

    a) $m(H, 0) = 0$, $m(H, K) = \overline{m}(H, \sqrt{K})$ for some analytic function $\overline{m}(H, S)$ which furthermore satisfies $(\partial \overline{m} / \partial S)(H, 0) > 0$, or
    b) $m(H, 0) = 2H$, $m(H, K) = \overline{m}(H, \sqrt{K})$ for some analytic function $\overline{m}(H, S)$ which furthermore satisfies $(\partial \overline{m} / \partial S)(H, 0) < 0$.

The natural examples for cases a) in the definitions above are the Harmonic($m = K/H$) and Geometric ($m = \sqrt{K}$) mean curvatures. For cases b), take $m = 2H - K/H$ and $m = 2H - \sqrt{K}$.

5.1. $\mathcal{M}$-mean curvature lines near a parabolic line: the 1-regular, case a). Let 0 be a parabolic point of a C^r, $r \geq 6$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where

$$
\begin{aligned}
z(x, y) = &\frac{k}{2}y^2 + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{d}{2}x^2y + \frac{c}{6}y^3 + \frac{A}{24}x^4 + \frac{B}{6}x^3y \\
&+ \frac{C}{4}x^2y^2 + \frac{D}{6}xy^3 + \frac{E}{24}y^4 + \sum_{i+j=5} r_{ij}\frac{x^i y^j}{i!j!} + O(6)
\end{aligned}
\tag{8}
$$

The coefficients of the first and second fundamental forms are given by:

$$
\begin{aligned}
E(x,y) =& 1 + O(4) \\
F(x,y) =& \frac{1}{2}akx^2y + kdxy^2 + \frac{k}{b}y^3 + O(4) \\
G(x,y) =& 1 + k^2y^2 + kdx^2y + 2kbxy^2 + kcy^3 + 0(4) \\
e(x,y) =& ax + dy + \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2 + \frac{1}{6}r_{50}x^3 + \frac{1}{2}r_{41}x^2y \\
& + \frac{1}{2}(r_{32} - ak^2)xy^2 + \frac{1}{6}(r_{23} - 3k^2d)y^3 + O(4) \\
f(x,y) =& dx + by + \frac{1}{2}Bx^2 + Cxy + \frac{1}{2}Dy^2 + \frac{1}{6}r_{41}x^3 + \frac{1}{2}r_{32}x^2y \\
& + \frac{1}{2}(r_{23} - k^2d)xy^2 + \frac{1}{6}(r_{14} - 3k^2b)y^3 + O(4) \\
g(x,y) =& k + bx + cy + \frac{1}{2}Cx^2 + Dxy + \frac{1}{2}(E - k^3)y^2 + \frac{1}{6}r_{32}x^3 \\
& + \frac{1}{2}(r_{23} - k^2d)x^2y + \frac{1}{2}(r_{14} - 3bk^2)xy^2 \\
& + \frac{1}{6}(r_{05} - 6k^2c)y^3 + O(4)
\end{aligned}
\tag{9}
$$

The Gaussian and the Mean curvatures are given by

$$
\begin{aligned}
\mathcal{K}(x,y) =& k(ax + dy) + \frac{1}{2}(Ak + 2ab - 2d^2)x^2 + (Bk + ac - bd)xy \\
& + \frac{1}{2}(Ck + 2cd - 2b^2)y^2 + \frac{1}{6}(kr_{50} + 3Ab + 3aC - 6Bd)x^3 \\
& + \frac{1}{2}(kr_{32} - 4ak^3 + 2cB + aE - 3bC)xy^2 \\
& + \frac{1}{2}(kr_{41} + cA + 2aD - 3Cd)x^2y \\
& + \frac{1}{6}(kr_{23} + 3cC + 3Ed - 6bD - 12k^3d)y^3 + O(4), \\
\mathcal{H}(x,y) =& \frac{1}{2}k + \frac{1}{2}(a + b)x + \frac{1}{2}(c + d)y + \frac{1}{4}(A + C)x^2 \\
& + \frac{1}{2}(B + D)xy + \frac{1}{4}(E + C - 3k^3)y^2 + \frac{1}{12}(r_{50} + r_{32})x^3 \\
& + \frac{1}{4}(r_{32} + r_{14} + k^2(2d - 9b - a))xy^2 \\
& + \frac{1}{4}(r_{41} + r_{23} + k^2(a - 3d))x^2y \\
& + \frac{1}{12}(r_{23} + r_{05} + k^2(3b - 9c - 3d))y^3 + O(4)
\end{aligned}
\tag{10}
$$

Let $\mathcal{M}$ be a 1- regular function. Write

$$\mathcal{M}(x,y) = m(\mathcal{H},\mathcal{K})(x,y) = (m_0 + m_1 x + m_2 y + O(2))\mathcal{K}(x,y).$$

The coefficients of the quadratic differential equation (1) are given by L, M, N as follows:

$$L = g - \mathcal{M}G, \quad M = 2(f - \mathcal{M}F), \quad N = e - \mathcal{M}E.$$

$$
\begin{aligned}
L =&\, k + (b - akm_0)x + (c - km_0d)y \\
&+ \frac{1}{2}[C + (2d^2 - 2ab - Ak)m_0 - 2akm_1]x^2 \\
&+ [D + (bd - ac - kB)m_0 - km_1d - akm_2]xy \\
&+ \frac{1}{2}[(2b^2 - kC - 2cd)m_0 - k^3 + E - 2km_2d]y^2 + O(3) \\
M =&\, 2dx + 2by + Bx^2 + 2Cxy + Dy^2 + O(3) \hspace{2cm} (11) \\
N =&\, a(1 - km_0)x + d(1 - km_0)y \\
&+ \frac{1}{2}[(2d^2 - 2ab - Ak)m_0 + A - 2akm_1]x^2 \\
&+ [B + (bd - ac - kB)m_0 - km_1d - akm_2]xy \\
&+ \frac{1}{2}[(2b^2 - 2cd - kC)m_0 - km_2d + C]y^2 + O(3)
\end{aligned}
$$

Lemma 3. *Let 0 be a parabolic point and consider the parametrization $(x, y, z(x, y))$ as above. If $k > 0$ and $a^2 + d^2 \neq 0$ then the set of parabolic points is locally a regular curve normal to the vector (a, d) at 0.*

> i) *If $a \neq 0$ the parabolic curve is transversal to the minimal principal direction $(1, 0)$.*
> ii) *If $a = 0$ then the parabolic curve is tangent to the principal direction given by $(1, 0)$ and has quadratic contact with the corresponding minimal principal curvature line if $dk(Ak - 3d^2) \neq 0$.*

Proof. If $a \neq 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $x = -\frac{d}{a}y + O_1(2)$ and so is transversal to the principal direction $(1, 0)$ at $(0, 0)$.

If $a = 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $y = \frac{2d^2 - Ak}{2dk}x^2 + O_2(3)$ and that $y = -\frac{d}{2k}x^2 + O_3(3)$ is the principal line tangent to the principal direction $(1, 0)$. Now the condition of quadratic contact $\frac{2d^2 - Ak}{2dk} \neq -\frac{d}{2k}$ is equivalent to $dk(Ak - 3d^2) \neq 0$. $\square$

Proposition 4. *Let 0 be a parabolic point and the Monge chart (x, y) as above. Suppose $\mathcal{M}$ is $1-$regular at $(k, 0)$ with $\partial m/\partial K(k, 0) = m_0 > 1/k$.*

> i) *If $a \neq 0$ then the mean $\mathcal{M}$-curvature lines are transversal to the parabolic curve and the mean curvatures lines are shown in the Figure 2, the cuspidal case.*

ii) *If $a = 0$ and $\sigma = (Ak - 3d^2) \neq 0$ then the mean $\mathcal{M}$- curvature lines are shown in the Figure 2. In fact, if $\sigma > 0$ then the $\mathcal{M}$-mean curvature lines are folded saddles. Otherwise, if $\sigma < 0$ then the $\mathcal{M}$-mean curvature lines are folded nodes or folded focus according to $\delta = [d^2(km_0 - 25) + 8Ak]$ be positive or negative. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrated in the Figure 2 below, are called parabolic separatrices.*
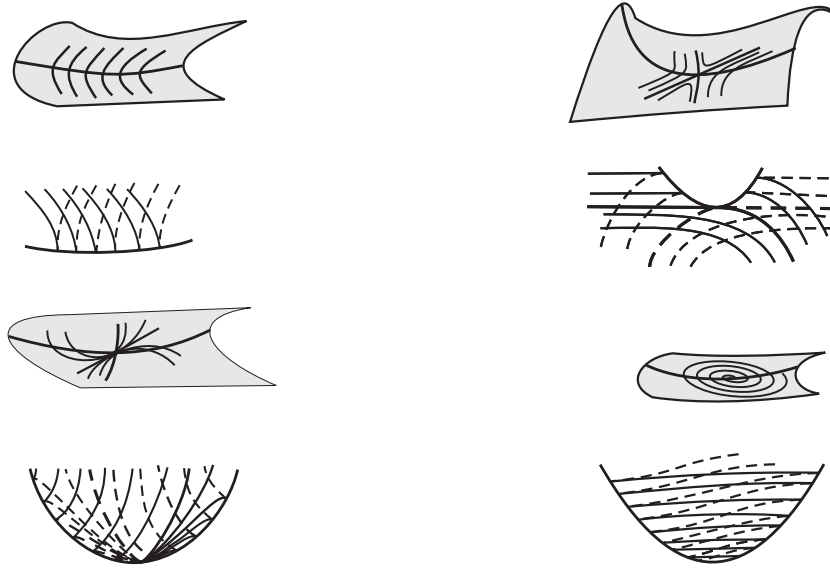

FIGURE 2. $\mathcal{M}$-mean curvature lines near a parabolic point (cuspidal, folded saddle, folded node and folded focus) and their separatrices

Proof. Consider the quadratic differential equation

$$H(x, y, [dx : dy]) = Ldy^2 + Mdxdy + Ndx^2 = 0$$

and the Lie-Cartan line field X of class C^{r-3} defined by

$$x' = H_p$$
$$y' = pH_p$$
$$p' = -(H_x + pH_y), \qquad p = \frac{dy}{dx}$$

where L, M and N are given by equation 11.

If $a \neq 0$ the vector Y is regular and therefore the $\mathcal{M}$-mean curvature lines are transversal to the parabolic line and at parabolic points these lines are tangent to the principal direction $(1, 0)$.

If $a = 0$, direct calculation gives $H(0) = 0,\quad H_x(0) = 0,\quad H_y(0) = -kd,\ \ H_p(0) = 0$.

$$DX(0) = \begin{pmatrix} 2d & 2b & 2k \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{12}$$

where,

$a_{31} = (Ak - 2d^2)m_0 - A,\ a_{32} = (kB - bd)m_0 - B + kdm_1+,\ a_{33} = (km_0 - 3)d$.

The non vanishing eigenvalues of $DX(0)$ are

$$\lambda_1 = \frac{1}{2}\Big[\, d(km_0 - 1) - \sqrt{(-1 + km_0)(d^2(km_0 - 25) + 8Ak)}\,\Big],$$

$$\lambda_2 = \frac{1}{2}\Big[\, d(km_0 - 1) + \sqrt{(-1 + km_0)(d^2(km_0 - 25) + 8Ak)}\,\Big]$$

Therefore, $\lambda_1\lambda_2 = 2(1 - km_0)(Ak - 3d^2)$.

It follows that 0 is a hyperbolic singularity provided $\sigma = (Ak - 3d^2) \neq 0$. If $\sigma > 0$ then the $\mathcal{M}$-mean curvature lines are folded saddles and if $\sigma < 0$ then the $\mathcal{M}$-mean curvature lines are folded nodes ($[d^2(km_0 - 25) + 8Ak] > 0$) or folded focus ($[d^2(km_0 - 25) + 8Ak] > 0$). See Figure 2. $\square$

Theorem 5. Assume that the $\mathcal{M}$-mean curvature function $\mathcal{M}$ is 1-regular as in definition 3, case a). An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 6$, is C^6-local $\mathcal{M}$-mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma\delta \neq 0$ in proposition 4 holds.

Proof. Direct from Lemma 3 and proposition 4, the local topological character of the foliation can be changed by small perturbation of the immersion when $\delta\sigma = 0$. $\square$

5.2. $\mathcal{M}$-mean curvature lines near a parabolic line: the 1/2-regular, case a).

Let 0 be a parabolic point of a C^r, $r \geq 6$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where z is as in equation 8.

The coefficients of the first and second fundamental forms are given by expressions 9.

The Gaussian and Arithmetic Mean curvatures are given by equations 10.

Below is established the typical behavior of $\mathcal{M}$- mean curvature lines for a function $\mathcal{M} = \mathcal{M}_1\sqrt{\mathcal{K}}$, as in definition 4.

Squaring both members of the differential equation $k_n(x, y, [dx : dy]) = \mathcal{M}_1\sqrt{\mathcal{K}}$ to remove the square root singularity, gives the following quartic differential equation:

$$A_{40}dx^4 + A_{31}dx^3dy + A_{22}dx^2dy^2 + A_{13}dxdy^3 + A_{04}dy^4 = 0 \tag{13}$$

where,

$$A_{40} = e^2(EG - F^2) - E^2(eg - f^2)\mathcal{M}_1^2$$
$$A_{31} = 4ef(EG - F^2) - 4EF(eg - f^2)\mathcal{M}_1^2$$
$$A_{22} = (4f^2 + 2eg)(EG - F^2) - (2EG + 4F^2)(eg - f^2)\mathcal{M}_1^2$$
$$A_{13} = 4fg(EG - F^2) - 4FG(eg - f^2)\mathcal{M}_1^2$$
$$A_{04} = g^2(EG - F^2) - G^2(eg - f^2)\mathcal{M}_1^2$$

Writing

$$\mathcal{M}_1(x,y) = m(\mathcal{H}(x,y), \mathcal{K}(x,y)) = m_0 + m_1 x + m_2 y + O(2),$$

the coefficients of the quartic differential equation (13) are given by

$$
\begin{aligned}
A_{40} = & -km_0^2(ax + dy) + \frac{1}{2}[2a^2 + m_0^2(2d^2 - 2ab - Ak) - 4akm_0 m_1]x^2 \\
& + [2ad - (Bk + bd - ac)m_0^2 - 2km_0(dm_1 + am_2)]xy \\
& + \frac{1}{2}[2d^2 + m_0^2(2b^2 - 2cd - Ck) - 4kdm_0 m_2]y^2 + 0(3) \\
A_{31} = & \, 4dax^2 + 4(d^2 + ab)xy + 4dby^2 + 0(3) \\
A_{22} = & \, 2k(1 - m_0^2)(ax + dy) \\
& + [m_0^2(2d^2 - 2ab - kA) - 4akm_0 m_1 + Ak + 2ab + 4d^2]x^2 \\
& + [10bd + 2ac + 2kB - 4km_0(m_1 d + am_2) + 2m_0^2(bd - kB - ac)]xy \\
& + [(2b^2 - kC - 2cd)m_0^2 - 4m_0 m_2 kd + kC + 2cd + 4b^2]y^2 + 0(3) \\
A_{13} = & \, 4k(dx + by) + (2Bk + 4bd)x^2 + 4(Ck + cd + b^2)xy \\
& + (2kD + 4bc)y^2 + 0(3) \\
A_{04} = & \, k^2 + k(2b - am_0^2)x + k(2c - dm_0^2)y \\
& + \frac{1}{2}[2b^2 + 2kC - 4akm_0 m_1 + m_0^2(2d^2 - 2ab - Ak)]x^2 \\
& + [2kD + 2bc + m_0^2(bd - kB - ac) - 2km_0(m_1 d + m_2 a)]xy \\
& + \frac{1}{2}[2c^2 - 2k^4 + 2kE + m_0^2(2b^2 - kC - 2cd) - 4m_0 m_2 d]y^2 + 0(3)
\end{aligned}
$$
$$(14)$$

Lemma 4. *Let* 0 *be a parabolic point and consider the parametrization* $(x, y, z(x,y))$ *as above. If* $k > 0$ *and* $a^2 + d^2 \neq 0$ *then the set of parabolic points is locally a regular curve normal to the vector* (a, d) *at* 0.

i) *If* $a \neq 0$ *the parabolic curve is transversal to the minimal principal direction* $(1, 0)$.

ii) *If* $a = 0$ *then the parabolic curve is tangent to the principal direction given by* $(1, 0)$ *and has quadratic contact with the corresponding minimal principal curvature line if* $dk(Ak - 3d^2)m_0^2 \neq 0$.

Proof. If $a \neq 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $x = -\frac{d}{a}y + O_1(2)$ and so is transversal to the principal direction $(1,0)$ at $(0,0)$.

If $a = 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $y = \frac{2d^2 - Ak}{2dk}x^2 + O_2(3)$ and that $y = -\frac{d}{2k}x^2 + O_3(3)$ is the principal line tangent to the principal direction $(1,0)$. Now the condition of quadratic contact $\frac{2d^2 - Ak}{2dk} \neq -\frac{d}{2k}$ is equivalent to $dk(Ak - 3d^2) \neq 0$. $\square$

Proposition 5. *Let 0 be a parabolic point and the Monge chart (x,y) as above and $\mathcal{M}$ be a mean curvature function 1/2-regular, case a).*

  i) *If $a \neq 0$ then the $\mathcal{M}$-mean curvature lines are transversal to the parabolic curve, as shown in Figure 3, the cuspidal case.*
  ii) *If $a = 0$ and $\sigma = dk(Ak - 3d^2)m_0^2 \neq 0$ then the $\mathcal{M}$- mean curvature lines are shown in the Figure 3. In fact, if $\sigma > 0$ then the $\mathcal{M}$- mean curvature lines are folded saddles. Otherwise, if $\sigma < 0$ then the $\mathcal{M}$- mean curvature lines are folded nodes. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrated in Figure 3 are called parabolic separatrices.*
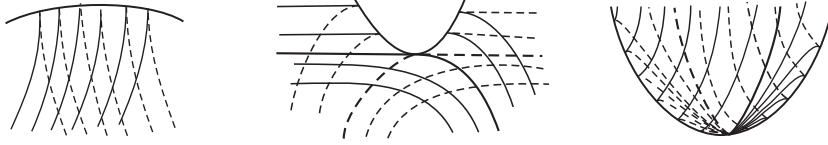

FIGURE 3. $\mathcal{M}$-mean curvature lines near a parabolic point (cuspidal, folded saddle and folded node) and their separatrices

Proof. Consider the quartic differential equation,

$$H(x,y,\frac{dy}{dx}) = A_{04}(\frac{dy}{dx})^4 + A_{13}(\frac{dy}{dx})^3 + A_{22}(\frac{dy}{dx})^2 + A_{31}\frac{dy}{dx} + A_{40} = 0$$

and the Lie-Cartan line field of class C^{r-3} defined by

$$x' = H_p$$
$$y' = pH_p$$
$$p' = -(H_x + pH_y), \qquad p = \frac{dy}{dx}$$

where A_{ij} are given by equation 14.

If $a \neq 0$ the vector Y is regular and therefore the $\mathcal{M}$-mean curvature lines are transversal to the parabolic curve. If $a = 0$, the parabolic curve is tangent to the principal direction $(1,0)$.

For $a = 0$, direct calculation gives $H(0) = 0$, $H_x(0) = 0$, $H_y(0) = -kd$, $H_p(0) = 0$.

Therefore, solving the equation $H(x, y(x, p), p) = 0$ near 0 it follows, by the Implicit Function Theorem, that:

$$y = y(x, p) = \frac{2d^2 - Ak}{2kd}x^2 - \frac{r_{50}k^2d + 6Akbd - 3BAk^2 - 6d^3b}{6k^2d^2}x^3 + O(4).$$

Therefore the vector field Y given by the differential equation below

$$x' = H_p(x, y(x, p), p)$$
$$p' = -(H_x + pH_y)(x, y(x, p), p)$$

is given by

$$x' = \frac{4d^3}{k}x^3 + 12d^2x^2p + 12kdxp^2 + 4k^2p^3 + O(4)$$
$$p' = (Ak - 2d^2)m_0^2x + kdm_0^2p + O(2).$$

The singular point 0 is isolated and the eigenvalues of the linear part of Y are given by $\lambda_1 = 0$ and $\lambda_2 = m_0^2kd$. The correspondent eigenvectors are given by $f_1 = (1, (2d^2 - Ak)/dk)$ and $f_2 = (0, 1)$.

Performing the calculations, restricting Y to the center manifold W^c of class C^{r-3}, $T_0W^c = f_1$, it follows that

$$Y_c = -\frac{2}{3}\frac{(Ak - 3d^2)^3}{kd^3}x^3 + 0(4)$$

It follows that 0 is a topological saddle or node of cubic type provided $\sigma(Ak - 3d^2)km_0^2d \neq 0$. If $\sigma > 0$ then the $\mathcal{M}$- mean curvature lines are folded saddles and if $\sigma < 0$ then the $\mathcal{M}$- mean curvature lines are folded nodes. In the case $\sigma > 0$, the center manifold W^c is unique, [31], cap. V, page 319, and so the saddle separatrices are well defined. See Figure 4 below.
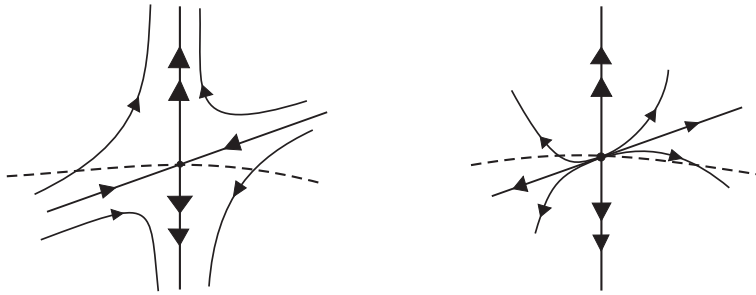

FIGURE 4. Phase portrait of the vector field Y near singularities

Notice that due to the constrains of the problem treated here, the non hyperbolic saddles and nodes, which in the standard theory would bifurcate into three singularities, are actually structurally stable (do not bifurcate). □

Remark 5. *The reader may find a more complete study of the partial hyperbolicity structure in the theorem above, which can be expressed in the context of normal hyperbolicity, in the paper of Palis and Takens* [25].

For a deeper analysis of the lost of the hyperbolicity condition and the consequent bifurcations, the reader is addressed to the book of Roussarie [28].

Remark 6. *As in the analysis of geometric mean curvature lines near a parabolic point, see* [12], *the singularity of the Lie-Cartan vector field above is semi-hyperbolic of order* 3. *However, the terms of third order of the functions A_{ij} of equation (14) have no contribution to the orbit structure around the singularity. This was confirmed by computer algebraic calculations.*

Theorem 6. Assume that the $\mathcal{M}$-mean curvature function $\mathcal{M}$ is 1/2-regular as in definition 4, case a).

An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 6$, is C^6-local $\mathcal{M}$- mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma \neq 0$ in proposition 5 holds.

Proof. Direct from Lemma 4 and proposition 5, the local topological character of the foliation can be changed by small perturbation of the immersion when $\sigma = 0$. □

5.3. $\mathcal{M}$- mean curvature lines near a parabolic line: the 1-regular, case b).

Let 0 be a parabolic point of a C^r, $r \geq 6$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where

$$
\begin{aligned}
z(x,y) =& \frac{k}{2}x^2 + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{d}{2}x^2y + \frac{c}{6}y^3 + \frac{A}{24}x^4 + \frac{B}{6}x^3y \\
&+ \frac{C}{4}x^2y^2 + \frac{D}{6}xy^3 + \frac{E}{24}y^4 + \sum_{i+j=5} r_{ij}\frac{x^iy^j}{i!j!} + O(6)
\end{aligned}
\tag{15}
$$

The coefficients of the first and second fundamental forms are given by:

$$E(x,y) = 1 + k^2 x^2 + kax^3 + 2kdx^2 y + kbxy^2 + O(4)$$

$$F(x,y) = \frac{1}{2}kdx^3 + kbx^2 y + \frac{1}{2}kcxy^2 + O(3)$$

$$G(x,y) = 1 + O(4)$$

$$e(x,y) = k + ax + dy + \frac{1}{2}(A - k^3)x^2 + Bxy + \frac{1}{2}Cy^2$$
$$+ \frac{1}{6}(r_{50} - 6ak^2)x^3 + (r_{41} - 3k^2 d)x^2 y$$
$$+ \frac{1}{2}(r_{32} - k^2 b)xy^2 + \frac{1}{6}r_{23}y^3 + O(4)$$

$$f(x,y) = dx + by + \frac{1}{2}Bx^2 + Cxy + \frac{1}{2}Dy^2 \qquad (16)$$
$$+ \frac{1}{6}(r_{41} - 3dk^2)x^3 + (r_{32} - k^2 b)x^2 y$$
$$+ \frac{1}{2}r_{23}xy^2 + \frac{1}{6}r_{14}y^3 + O(4)$$

$$g(x,y) = bx + cy + \frac{1}{2}Cx^2 + Dxy + \frac{1}{2}Ey^2$$
$$+ \frac{1}{6}(r_{32} - 3k^2 b)x^3 + \frac{1}{2}(r_{23} - k^2 c)x^2 y$$
$$+ \frac{1}{2}r_{14}xy^2 + \frac{1}{6}r_{05}y^3 + O(4)$$

The Gaussian and the Mean curvatures are given by

$$
\begin{aligned}
\mathcal{K}(x,y) =\,& k(bx + cy) + \frac{1}{2}(2ab + kC - 2d^2)x^2 + (ac - bd + kD)xy \\
&+\frac{1}{2}(2cd + kE - 2b^2)y^2 + \frac{1}{6}(3Ab - 12k^3b + 3aC + kr_{32} - 6dB)x^3 \\
&+\frac{1}{2}(2aD - 4k^3c - 3dC + Ac + r_{23}k)x^2y \\
&+\frac{1}{2}(2Bc + aE - 3bC + r_{14}k)xy^2 \\
&+\frac{1}{6}(3cC - 6bD + 3dE + kr_{05})y^3 + O(4), \\
\mathcal{H}(x,y) =\,& \frac{1}{2}k + \frac{1}{2}(b + a)x + \frac{1}{2}(c + d)y + \frac{1}{4}(C + A - 3k^3)x^2 \\
&+\frac{1}{2}(D + B)xy + \frac{1}{4}(E + C)y^2 + \frac{1}{12}(r_{50} + r_{32} - 3k^2(b + 6a))x^3 \\
&+\frac{1}{4}(r_{41} + r_{23} - k^2(c + 9d))x^2y + \frac{1}{4}(r_{32} + r_{14} - 3k^2b)xy^2 \\
&+\frac{1}{12}(r_{23} + r_{05})y^3 + O(4)
\end{aligned}
$$

$$(17)$$

Let $\mathcal{M}$ be a 1- regular function. Write

$$
\mathcal{M}(x,y) = m(\mathcal{H}, \mathcal{K})(x,y) = 2\mathcal{H} - (m_0 + m_1x + m_2y + O(2))\mathcal{K}(x,y).
$$

The coefficients of the quadratic differential equation (1) are given by L, M, N as follows:

$$
L = g - \mathcal{M}G, \quad M = 2(f - \mathcal{M}F), \quad N = e - \mathcal{M}E.
$$

$$
\begin{aligned}
L =\,& -k + (kbm_0 - a)x + (kcm_0 - d)y \\
&+\frac{1}{2}[m_0(kC + 2ab - 2d^2) + 2kbm_1 + 3k^3 - A]x^2 \\
&+[m_0(ac + kD - bd) + k(cm_1 + bm_2) - B]xy \\
&+\frac{1}{2}[m_0(kE + 2cd - 2b^2) - C + 2kcm_2]y^2 + O(3) \\
M =\,& 2dx + 2by + Bx^2 + 2Cxy + Dy^2 + O(3) \\
N =\,& (-1 + km_0)(bx + cy) + \frac{1}{2}[m_0(2ab + kC - 2d^2) + 2kbm_1 - C]x^2 \\
&+[m_0(ac + kD - bd) + k(bm_2 + cm_1) - D]xy \\
&+\frac{1}{2}[m_0(2cd - 2b^2 + kE) + 2kcm_2 - E]y^2 + O(3)
\end{aligned}
$$

$$(18)$$

Lemma 5. *Let 0 be a parabolic point and consider the parametrization $(x, y, z(x, y))$ as above. If $k > 0$ and $b^2 + c^2 \neq 0$ then the set of parabolic points is locally a regular curve normal to the vector (b, c) at 0.*

   i) *If $b \neq 0$ the parabolic curve is transversal to the maximal principal direction $(1, 0)$.*
   ii) *If $b = 0$ then the parabolic curve is tangent to the maximal principal direction given by $(1, 0)$ and has quadratic contact with the corresponding maximal principal curvature line if $d(2d - c) - kC \neq 0$.*

Proof. If $b \neq 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $y = -\frac{b}{c}x + O_1(2)$ and so is transversal to the maximal principal direction $(1, 0)$ at $(0, 0)$.

If $b = 0$, from the expression of $\mathcal{K}$ given by equation 10 it follows that the parabolic line is given by $y = \frac{2d^2 - Ck}{2ck}x^2 + O_2(3)$ and that $y = \frac{d}{2k}x^2 + O_3(3)$ is the principal line tangent to the principal direction $(1, 0)$. Now the condition of quadratic contact $\frac{2d^2 - kC}{2ck} \neq \frac{d}{2k}$ is equivalent to $d(2d - c) - kC \neq 0$. $\square$

Proposition 6. *Let 0 be a parabolic point and the Monge chart (x, y) as above. Suppose $\mathcal{M}$ is $1-$regular, case b), at $(k, 0)$ with $\partial m / \partial K(k, 0) = -m_0 < -1/k < 0$.*

   i) *If $b \neq 0$ then the $\mathcal{M}$- mean curvature lines are transversal to the parabolic curve and the mean curvatures lines are shown in the Figure 2, the cuspidal case.*
   ii) *If $b = 0$ and $\sigma = (cd - 2d^2 + kC)kcm_0^2 \neq 0$ then the $\mathcal{M}$- mean curvature lines are shown in the Figure 2. In fact, if $\sigma > 0$ then the $\mathcal{M}$-mean curvature lines are folded saddles. Otherwise, if $\sigma < 0$ then the mean $\mathcal{M}$- curvature lines are folded nodes or folded focus according to $\delta = (km_0 - 1)(c^2 + 8kC + 8d(c - 2d))$ be positive or negative. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrated in the Figure 2 of proposition 4, are called parabolic separatrices.*

Proof. Consider the quadratic differential equation

$$H(x, y, [dx : dy]) = Ldy^2 + Mdxdy + Ndx^2 = 0$$

and the Lie-Cartan line field X of class C^{r-3} defined by

$$x' = H_p$$
$$y' = pH_p$$
$$p' = -(H_x + pH_y), \qquad p = \frac{dy}{dx}$$

where L, M and N are given by equation 18.

If $b \neq 0$ the vector Y is regular and therefore the $\mathcal{M}$- mean curvature lines are transversal to the parabolic line and at parabolic points these lines are tangent to the principal direction $(1, 0)$.

If $b = 0$, direct calculation gives $H(0) = 0$, $H_x(0) = 0$, $H_y(0) = c(km_0 - 1)$, $H_p(0) = 0$.

$$DX(0) = \begin{pmatrix} 2d & 0 & -2k \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{19}$$

where,

$$a_{31} = C + m_0(2d^2 - kC), \quad a_{32} = D - (kD + ac)m_0 - kcm_1,$$
$$a_{33} = c - 2d - kcm_0.$$

The non vanishing eigenvalues of $DX(0)$ are

$$\lambda_1 = \frac{1}{2}\left[\, c(1 - km_0) - \sqrt{(-1 + km_0)(km_0c^2 - c^2 + 8kC + 8d(c - 2d))}\,\right],$$

$$\lambda_2 = \frac{1}{2}\left[\, c(1 - km_0) + \sqrt{(-1 + km_0)(km_0c^2 - c^2 + 8kC + 8d(c - 2d))}\,\right]$$

Therefore, $\lambda_1\lambda_2 = 2(1 - km_0)(kC + cd - 2d^2)$.

It follows that 0 is a hyperbolic singularity provided $\sigma = (cd - 2d^2 + kC)kcm_0^2 \neq 0$. If $\sigma > 0$ then the $\mathcal{M}$-mean curvature lines are folded saddles and if $\sigma < 0$ then the $\mathcal{M}$-mean curvature lines are folded nodes ($(c^2 + 8kC + 8d(c - 2d)) > 0$) or folded focus ($(c^2 + 8kC + 8d(c - 2d)) < 0$). $\qquad\square$

Theorem 7. Assume that the $\mathcal{M}$-mean curvature function $\mathcal{M}$ is 1-regular as in definition 3, case b). An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 6$, is C^6–local $\mathcal{M}$-mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma\delta \neq 0$ in proposition 4 holds.

Proof. Direct from Lemma 4 and proposition 6, the local topological character of the foliation can be changed by small perturbations of the immersion when $\delta\sigma = 0$. $\qquad\square$

5.4. $\mathcal{M}$- mean curvature lines near a parabolic line: the 1/2-regular, case b).

Below are formulated two results describing the typical behavior and genericety of $\mathcal{M}$- mean curvature lines for a function $\mathcal{M} = 2\mathcal{H} - \mathcal{M}_1\sqrt{\mathcal{K}}$.

Proposition 7. *Let* 0 *be a parabolic point and the Monge chart* (x, y) *as in equation 15. Suppose that* $\mathcal{M}$ *is a mean curvature function 1/2-regular, case b).*

    i) *If* $b \neq 0$ *then the* $\mathcal{M}$-*mean curvature lines are transversal to the parabolic curve, as shown in Figure 3, the cuspidal case.*

    ii) *If* $b = 0$ *and* $\sigma = (cd - 2d^2 + kC)kcm_0^2 \neq 0$ *then the* $\mathcal{M}$-*mean curvature lines are shown in the Figure 3. In fact, if* $\sigma > 0$ *then the* $\mathcal{M}$-*mean curvature lines are folded saddles. Otherwise, if* $\sigma < 0$ *then the* $\mathcal{M}$- *mean curvature lines are folded nodes.*

Proof. Similar to that of proposition 5. □

Theorem 8. Assume that the $\mathcal{M}$- mean curvature function $\mathcal{M}$ is $1/2$-regular as in definition 4, case b).

An immersion $\alpha \in \mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq 6$, is C^6-local $\mathcal{M}$- mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma \neq 0$ in proposition 7 holds.

Proof. Similar to that of theorem 5. □

6. ON $\mathcal{M}$- MEAN CURVATURE STRUCTURAL STABILITY

In this section the results of sections 3, 4 and 5 are put together to provide sufficient conditions for $\mathcal{M}$-mean curvature stability.

Theorem 9. Let $\mathcal{M}$ be a mean curvature function which is positive regular, 1-regular or $1/2$-regular. See definitions 2, 3 and 4.

Then the set of immersions $\mathcal{G}_i(\mathbb{V}^2), i = 1,\ 2$ which satisfy conditions i), ... , v) below are i-C^s-$\mathcal{M}$-mean curvature structurally stable and $\mathcal{G}_i, i = 1,\ 2$ is open in $\mathcal{I}^{r,s}(\mathbb{V}^2)$, $r \geq s \geq 6$.

  i) The parabolic curve is 1-regular or $1/2$-regular: $\mathcal{K} = 0$ implies $d\mathcal{K} \neq 0$ and the tangential singularities are folded saddles, nodes and foci.
  ii) The umbilic points are of type M_i, $i = 1,\ 2,\ 3$.
  iii) The cycles of $\mathbb{H}^{\mathcal{M}}_{\alpha,i}$ are hyperbolic.
  iv) The foliations $\mathbb{H}^{\mathcal{M}}_{\alpha,i}$ have no separatrix connections. This means that there is no $\mathcal{M}$-mean curvature line joining two umbilic or tangential parabolic singularities and being separatrices at both ends. See propositions 1, 4, 5, 6 and 7.
  v) The limit set of every leaf of $\mathbb{H}^{\mathcal{M}}_{\alpha,i}$ is a parabolic point, umbilic point or a $\mathcal{M}$-mean curvature cycle.

Proof. The openness of $\mathcal{G}_i(\mathbb{V}^2)$ follows from the local structure of the $\mathcal{M}$-mean curvature lines near the umbilic points of types M_i, $i = 1, 2, 3$, near the parabolic points (cusp, saddles, foci and nodes), near the $\mathcal{M}$-mean curvature cycles and by the absence of umbilic $\mathcal{M}$-mean curvature separatrix connections and the absence of recurrences. The equivalence can be performed by the method of canonical regions and their continuation as was done in [16, 18] for principal lines, and in [14], for asymptotic lines. □

Notice that Theorem 9 can be reformulated so as to give the $\mathcal{M}$-mean stability of the configuration rather than that of the separate foliations. To this end it is necessary to consider the folded extended lines, that is to consider the line of one foliation that arrive at the parabolic set at a given transversal point as continuing through the line of the other foliation leaving the parabolic set at this point, in a sort of "billiard". This gives raise to the extended folded cycles and separatrices that must be preserved by the homeomorphism mapping simultaneously the two foliations.

Therefore the third, fourth and fifth hypotheses above should be modified as follows:

  iii') the extended folded periodic cycles should be hyperbolic,
  iv') the extended folded separatrices should be disjoint,
   v') the limit set of extended lines should be umbilic points, parabolic singularities and extended folded cycles.

The class of immersions which verify the extended five conditions i), ii), iii'), iv'), v') of a compact and oriented manifold $\mathbb{V}^2$ will be denoted by $\mathcal{G}(\mathbb{V}^2)$.

This procedure has been adopted by the authors in the case of asymptotic lines by the suspension operation in order to pass from the foliations to the configuration and properly formulate the stability results. See [14].

Remark 7. *In the space of convex immersions $\mathcal{I}_c^{r,s}(\mathbb{S}^2)$ ($\mathcal{K}_\alpha > 0$), the sets $\mathcal{G}(\mathbb{S}^2)$ and $\mathcal{G}_1(\mathbb{S}^2) \cap \mathcal{G}_2(\mathbb{S}^2)$ coincide.*

The density result involving the five conditions above is formulated now.

Theorem 10. Let $\mathcal{M}$ be a mean curvature function which is positive regular, 1-regular or 1/2-regular. See definitions 2, 3 and 4.
   Then the sets $\mathcal{G}_i$, $i = 1,\ 2$, are dense in $\mathcal{I}^{r,2}(\mathbb{V}^2)$, $r \geq 6$.
   In the space $\mathcal{I}_c^{r,2}(\mathbb{S}^2)$ the set $\mathcal{G}(\mathbb{S}^2)$ is dense.

The main ingredients for the proof of this theorem are the Lifting and Stabilization Lemmas, essential for the achievement of condition five, are developed in next section.

7. Density of $\mathcal{M}$-Mean Curvature Structurally Stable Immersions

In this section will be proved an approximation theorem for the class of immersions or surfaces having structurally stable $\mathcal{M}$- *mean curvature configuration*.

The proof of Theorem 10 follows from the elimination of $\mathcal{M}$- mean curvature recurrences and the stabilization of the $\mathcal{M}$- mean curvature separatrices. The steps are basically those followed by C. Gutierrez and J. Sotomayor in the case of principal curvature lines, see [17, 18]. The main ideas goes back to M. Peixoto [26] and C. Pugh [27] to solve the similar problem of elimination of recurrences for vector fields on surfaces. See also the book by W. de Melo and J. Palis [23].

In what follows will be established the main technical lemmas necessary to obtain the Lifting Lemma, essential to control the effect on $\mathcal{M}$-mean curvature lines under suitable deformations of the immersion. There is no lost of generality to assume that the immersion is C^∞ or C^ω in the proof of the density theorem.

In what follows a chart whose coordinates lines are $\mathcal{M}$-mean curvature lines will be called a $\mathcal{M}$-*mean curvature chart* for the immersion α.

Lemma 6. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^∞ and $(u, v) :$ $(U, D) \to (V, I \times I)$ be a positive $\mathcal{M}$-mean curvature chart on $\mathbb{V}^2$, where $I = [-1, 1]$. Suppose that, for ϵ small, $\beta = \alpha_\epsilon = \alpha + \epsilon\varphi N$ is an immersion and φ be a smooth function on U which satisfies: $\varphi(-1, v) = \varphi(1, v) = \varphi_u(-1, v) = \varphi_u(1, v) = \varphi_{uu}(-1, v) = \varphi_{uu}(1, v) = 0$. Then the $\mathcal{M}$-mean curvature line of α_ϵ on D which passes through q in $\{u = -1\} \cap \{-1 < v < 1\}$ meets the segment of abscissa $\{u = 1\}$ at a point whose $v-$coordinate v_ϵ has a derivative with respect to ϵ given by:*

$$\frac{d}{d\epsilon}(v_\epsilon)|_{\epsilon=0} = \int_{-1}^{1} \frac{E[2\mathcal{M}\mathcal{M}_K + \mathcal{M}_H]}{4\tau_g\sqrt{EG - F^2}}\varphi_{vv}du + \int_{-1}^{1} A_1(u)\varphi_v du$$
$$+ \int_{-1}^{1} A_2(u)\varphi du \tag{20}$$

where A_1 and A_2 are functions of the coefficients of the first and second fundamental form of α.

Proof. Suppose that for ϵ small,

$$\beta(u, v, \epsilon) = \alpha_\epsilon(u, v) = \alpha(u, v) + \epsilon\varphi(u, v)N(u, v)$$

is an immersion.

The v-coordinate, $v = v(u, q, \epsilon)$, of the point where the line of $\mathcal{M}$-mean curvature through the point q in $\{u = -1\} \cap \{-1 < v < 1\}$ meets the curve with abscissa $\{u\}$, satisfies the following Cauchy Problem with parameter ϵ.

$$(e - \mathcal{M}E) + 2(f - \mathcal{M}F)\frac{dv}{du} + (g - \mathcal{M}G)(\frac{dv}{du})^2 = 0, \quad v(-1, \epsilon) = q \tag{21}$$

Since (u, v) is a $\mathcal{M}$-mean curvature chart, it results that

$$\frac{dv}{du}(u, q, 0) = 0,$$
$$(e - \mathcal{M}E)(u, v, 0) = (g - \mathcal{M}G)(u, v, 0) = 0, \tag{22}$$
$$(f - \mathcal{M}F)(u, v, 0) = \tau_g\sqrt{EG - F^2}$$
$$= \sqrt{(k_2 - \mathcal{M})(\mathcal{M} - k_1)}\sqrt{EG - F^2} \neq 0$$

Differentiating the equation (21) with respect to ϵ, evaluated on $(u, v(q), \epsilon)$, making $\epsilon = 0$ and using (22) it follows that

$$\frac{dv_\epsilon}{d\epsilon} = \frac{\partial v_\epsilon}{\partial \epsilon}(u, q, \epsilon)|_{\epsilon=0}$$

satisfies the following Cauchy Problem:

$$\frac{d}{du}(\frac{dv_\epsilon}{d\epsilon}) = -\frac{[e_\epsilon - \mathcal{M}_\epsilon E - \mathcal{M}E_\epsilon]}{2(f - \mathcal{M}F)}(u, v(q), 0)$$
$$\frac{dv_\epsilon}{d\epsilon}(-1, q, 0) = 0 \tag{23}$$

The structure equations for the immersion α are given by:

$$
\begin{aligned}
N_u &= \frac{fF - eG}{EG - F^2}\alpha_u + \frac{eF - fE}{EG - F^2}\alpha_v \\
N_v &= \frac{gF - fG}{EG - F^2}\alpha_u + \frac{fF - gE}{EG - F^2}\alpha_v \\
\alpha_{uu} &= \Gamma_{11}^1\alpha_u + \Gamma_{11}^2\alpha_v + eN \\
\alpha_{uv} &= \Gamma_{12}^1\alpha_u + \Gamma_{12}^2\alpha_v + fN \\
\alpha_{vv} &= \Gamma_{22}^1\alpha_u + \Gamma_{22}^2\alpha_v + gN
\end{aligned}
\tag{24}
$$

The functions Γ_{ij}^k are the Christoffel symbols whose expressions in terms of E, F and G in a chart are (u, v) are given by:

$$
\begin{aligned}
\Gamma_{11}^1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, &\quad \Gamma_{11}^2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\
\Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG - F^2)}, &\quad \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG - F^2)}, \\
\Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, &\quad \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.
\end{aligned}
\tag{25}
$$

By direct calculation, it is obtained that

$$
\begin{aligned}
\beta_u &= (1 + \epsilon\varphi\frac{fF - eG}{EG - F^2})\alpha_u + \epsilon\varphi\frac{eF - fE}{EG - F^2}\alpha_v + \epsilon\varphi_u N \\
\beta_v &= \epsilon\varphi\frac{gF - fG}{EG - F^2}\alpha_u + (1 + \epsilon\varphi\frac{fF - gE}{EG - F^2})\alpha_v + \epsilon\varphi_v N
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\beta_{uu} =& [\Gamma_{11}^1 + \epsilon\varphi\big(\Gamma_{11}^1\frac{fF - eG}{EG - F^2} + \Gamma_{12}^1\frac{eF - fE}{EG - F^2} + (\frac{fF - eG}{EG - F^2})_u\big) \\
&+ 2\epsilon\varphi_u\frac{fF - eG}{EG - F^2}]\alpha_u \\
&+ [\Gamma_{11}^2 + \epsilon\varphi\big(\Gamma_{11}^2\frac{fF - eG}{EG - F^2} + \Gamma_{12}^2\frac{eF - fE}{EG - F^2} + (\frac{eF - fE}{EG - F^2})_u\big) \\
&+ 2\epsilon\varphi_u\frac{eF - fE}{EG - F^2}]\alpha_v \\
&+ [e + \epsilon\varphi\frac{2efF - e^2G - f^2E}{EG - F^2} + \epsilon\varphi_{uu}]N
\end{aligned}
\tag{27}
$$

$$\beta_{uv} = [\Gamma_{12}^1 + \epsilon\varphi\big(\Gamma_{12}^1 \frac{fF - eG}{EG - F^2} + \Gamma_{22}^2 \frac{eF - fE}{EG - F^2} + (\frac{fF - eG}{EG - F^2})_v\big)$$
$$+\epsilon\varphi_u \frac{gF - fG}{EG - F^2} + \epsilon\varphi_v \frac{fF - eG}{EG - F^2}]\alpha_u$$
$$+[\Gamma_{12}^2 + \epsilon\varphi\big(\Gamma_{12}^2 \frac{fF - eG}{EG - F^2} + \Gamma_{12}^2 \frac{eF - fE}{EG - F^2} + (\frac{eF - fE}{EG - F^2})_v\big) \qquad (28)$$
$$+\epsilon\varphi_u \frac{fF - gE}{EG - F^2} + \epsilon\varphi_v \frac{eF - fE}{EG - F^2}]\alpha_v$$
$$+[f + \epsilon\varphi(f\frac{fF - eG}{EG - F^2} + g\frac{eF - fE}{EG - F^2} + \epsilon\varphi_{uv}]N$$

$$\beta_{vv} = [\Gamma_{22}^1 + \epsilon\varphi\big(\Gamma_{22}^1 \frac{fF - gE}{EG - F^2} + \Gamma_{12}^1 \frac{gF - fG}{EG - F^2} + (\frac{gF - fG}{EG - F^2})_u\big)$$
$$+2\epsilon\varphi_v \frac{gF - fG}{EG - F^2}]\alpha_u$$
$$+[\Gamma_{22}^2 + \epsilon\varphi\big(\Gamma_{22}^2 \frac{fF - gE}{EG - F^2} + \Gamma_{12}^2 \frac{gF - fG}{EG - F^2} + (\frac{fF - gE}{EG - F^2})_v\big) \qquad (29)$$
$$+2\epsilon\varphi_v \frac{fF - gE}{EG - F^2}]\alpha_v$$
$$+[g + \epsilon\varphi(g\frac{fF - gE}{EG - F^2} + f\frac{gF - fG}{EG - F^2}) + \epsilon\varphi_{vv}]N$$

Also,
$$\frac{\partial}{\partial \epsilon}(|\beta_u \wedge \beta_v|)|_{\epsilon=0} = -2\varphi\mathcal{H}\sqrt{EG - F^2} \qquad (30)$$

Therefore, using the equations (26) - (29) the following is obtained.

$$E_\epsilon = -2\varphi e, \qquad F_\epsilon = -2\varphi f, \qquad G_\epsilon = -2\varphi g$$
$$e_\epsilon = \varphi_{uu} + \varphi[\frac{2efF - e^2G - f^2F}{EG - F^2}] - \varphi_u\Gamma_{11}^1 - \varphi_v\Gamma_{11}^2$$
$$f_\epsilon = \varphi_{uv} + \varphi[f\frac{fF - eG}{EG - F^2} + g\frac{eF - fE}{EG - F^2}] - \varphi_u\Gamma_{12}^1 - \varphi_v\Gamma_{12}^2 \qquad (31)$$
$$g_\epsilon = \varphi_{vv} + \varphi[g\frac{fF - gE}{EG - F^2} + f\frac{gF - fG}{EG - F^2}] - \varphi_u\Gamma_{22}^1 - \varphi_v\Gamma_{22}^2.$$

Let
$$\mathcal{H} = \frac{Eg + eG - 2fF}{2(EG - F^2)}, \qquad \mathcal{K} = \frac{eg - f^2}{EG - F^2}. \qquad (32)$$

Then, using equations (31) and (32), it follows that

$$\mathcal{K}_\epsilon(EG - F^2) = \varphi K_1 + \varphi_u(2\Gamma_{12}^1 f - g\Gamma_{11}^1 - e\Gamma_{22}^1)$$
$$+\varphi_v(2\Gamma_{12}^2 f - g\Gamma_{11}^2 - e\Gamma_{22}^1) + g\varphi_{uu} - 2f\varphi_{uv} + e\varphi_{vv} \qquad (33)$$
$$2\mathcal{H}_\epsilon(EG - F^2) = \varphi H_1 - \varphi_u\Gamma_{22}^1 - \varphi_v\Gamma_{22}^2 + \varphi_{vv}$$

where,

$$K_1 = K_1(e, f, g, E, F, G)(u, v, 0), \quad H_1 = H_1(e, f, g, E, F, G)(u, v, 0).$$

Now as $\mathcal{M} = m(\mathcal{H}, \mathcal{K})$ it follows, from equations (22) and (33), that

$$\frac{d}{d\epsilon}\mathcal{M}_\epsilon = \mathcal{M}_H \mathcal{H}_\epsilon + \mathcal{M}_K \mathcal{K}_\epsilon$$

$$= \frac{\varphi_{vv}}{EG - F^2}\left(\frac{E\mathcal{M}_H}{2} + E\mathcal{M}\mathcal{M}_K\right) + (.)\varphi + (.)\varphi_u + (.)\varphi_v + (.)\varphi_{uv}$$

Therefore,

$$\frac{d}{d\epsilon}\left(\frac{e - \mathcal{M}E}{2(f - \mathcal{M}F)}\right)\big|_{\epsilon=0} = \frac{E(\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K)}{4(EG - F^2)(f - \mathcal{M}F)}\varphi_{vv} \qquad (34)$$
$$+ (.)\varphi + (.)\varphi_u + (.)\varphi_v + (.)\varphi_{uv} + (.)\varphi_{uu}$$

Here (.) denote functions involving $\mathcal{M}(\mathcal{H}, \mathcal{K})$ and the coefficients of the first and second fundamental forms of α.

Using (34) when integrating the variational equation (23) and performing the partial integration with boundary conditions on the function φ, the expression for $(\frac{dv_\epsilon}{d\epsilon})|_{\epsilon=0}$ is achieved, as stated in (20). $\qquad \square$

Lemma 7. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^∞ and $(u, v) : (U, D) \to (V, I \times I)$ be a positive $\mathcal{M}$-mean curvature chart on $\mathbb{V}^2$, where $I = [-1, 1]$. Suppose that $\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K > 0$, i.e, $\mathcal{M}$ is positive regular. Then there exists a smooth function $\varphi : \mathbb{V}^2 \to [0, 1]$ whose support is contained in D such that, if ϵ is small enough then, for every ϵ in $[-r, r]$, $\beta = \alpha + \epsilon\varphi N$ is an immersion and the $\mathcal{M}$-mean curvature line for β on D which passes through q in $\{u = -1\} \cap \{-1 < v < 1\}$ meets the segment $\{u = 1\} \times \{-1 < v < 1\}$ at a point $v_\epsilon(q)$ so that the map $\epsilon \to v_\epsilon(q)$ is strictly increasing.*

Proof. Let ρ be a real smooth function with values in $[0, 1]$, identically equal to 1 on a neighborhood of 0 and with support contained in I.

Let $\varphi = \varphi(u, v) = b\frac{v^2}{2}\rho(u)\rho(v)$ and take $r > 0$ small so that for any ϵ in $[-r, r]$, $\beta = \alpha_\epsilon = \alpha + \epsilon\varphi N$ is a smooth immersion. Let $v(u)$, $u \in I$, be the v−coordinate of the $\mathcal{M}$-mean curvature lines of $\alpha_\epsilon = \alpha + \epsilon\varphi N$ such that $v_\epsilon(q) = q$. As $\varphi(u, 0) = \varphi_v(u, 0) = 0$ and $\varphi_{vv}(u, 0) = b\rho(u)$ by lemma 6 applied to the family of immersions α_ϵ it follows that

$$\frac{\partial v}{\partial \epsilon}(u, \epsilon)\big|_{(0,0)} = \int_{-1}^{1} \frac{E(\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K)}{4\tau_g\sqrt{EG - F^2}}\rho(u)du = c > 0.$$

This implies that the map $\epsilon \to v_\epsilon(q)$ is strictly increasing. This proves the lemma. $\qquad \square \qquad\qquad \square$

Lemma 8. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion and $(u, v) : (U, D) \to (V, I \times I)$ be a positive $\mathcal{M}$-mean curvature chart for α on $\mathbb{V}^2$, where $I = [-1, 1]$.*

Assume also that $\mathcal{M}$ is positive regular. Then given any $\eta > 0$, there are numbers $d, c \in (0, \frac{1}{12})$ such that for every $r \in (0, d]$ and q in $\{u = -1\} \cap \{-\frac{1}{2} < v < \frac{1}{2}\}$, there exists a smooth function $\varphi : \mathbb{V}^2 \to [0, 1]$ whose support is contained in $D_r = v^{-1}(v(q) + rI)$ and $\|\varphi\|_{2,V}$, the C^2-norm of φ on V, in the $(u, v)-$coordinate chart, is less than η.

Furthermore, for every $\epsilon \in I$, $\alpha_\epsilon = \alpha + \epsilon\varphi N$ is an immersion and the $\mathcal{M}$-mean curvature line for α_ϵ on D which passes through q in $\{u = -1\} \cap \{-1 < v < 1\}$ meets the segment $\{u = 1\} \cap \{-1 < v < 1\}$ at a point $v_\epsilon(q)$ so that the map $\epsilon \to v_\epsilon(q)$ is strictly increasing and its image contains the interval $[v(q) - 2c\epsilon, v(q) + 2c\epsilon]$.

Proof. Let ρ be a real smooth function with values in $[0, 1]$, identically equal to 1 on $\frac{5}{6}I$ and with support contained in $\frac{6}{7}I$. Let also $\eta > 0$ be given. There are real numbers $c > 0$ and b such that for all (u_0, v_0) in $I \times I$ it follows that,

$$6|b|(\|\rho\|_2)^2 < \eta \tag{35}$$

$$\int_{-1}^0 b\frac{E(\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K)(u, v_0)}{4\tau_g\sqrt{EG - F^2}}\rho(u)du < \frac{1}{4} \tag{36}$$

$$\int_{-1}^1 b\frac{E(\mathcal{M}_H + 2\mathcal{M}\mathcal{M}_K)(u, v_0)}{4\tau_g\sqrt{EG - F^2}}\rho(u)du > 3c \tag{37}$$

Let ψ be a smooth real function on $U \times I \times I$, defined by
$$\psi(u, v; v_0, \epsilon) = b\epsilon\frac{(v-v_0)^2}{2}\rho(u).$$
It will be proved that if $d = d(\eta) \in (0, \frac{1}{12})$ is small enough, then for every $\epsilon \in (0, d)$ and q with $u(q) = -1$ and $v(q) = v_0$ in $\frac{1}{2}I$, the smooth function $\varphi(.) = \varphi(.; v(q), \epsilon)$ defined on $\mathbb{V}^2$ by

$$\varphi(u, v; v_0, \epsilon) = \psi(u, v; v_0, \epsilon)\rho(\frac{v - v_0}{|\epsilon|})$$

whose support is contained in D_r, satisfies the conditions required by the lemma.

In fact, suppose $d > 0$ is so small that for any $(v_0, \epsilon) \in I \times dI$, $\alpha_{v_0,\epsilon} = \alpha + \psi(.; v_0, \epsilon)N$ is an immersion.

Let $v(u; v_0, \epsilon)$, $u \in I$ and $v \in \frac{4}{5}I$, be the $v-$coordinate of a $\mathcal{M}$-mean curvature line of α_ϵ through the point q, with $u(q) = -1$ and $v(q) = v_0$. As $\psi(u, v_0; v_0, \epsilon) = 0$, using (35), (36) and (37), it follows from lemmas 6 and 7, applied to the family of immersions $\alpha_{v_0,\epsilon}$, depending on the parameter ϵ, that for all (u, v_0) in $I \times \frac{3}{4}I$,
$$\frac{\partial v}{\partial \epsilon}(1, v_0, 0) > 2c \qquad \text{and} \qquad \frac{\partial v}{\partial \epsilon}(1, v_0, 0) < \tfrac{1}{3}.$$
Hence, as I is compact and $\frac{\partial v}{\partial \epsilon}(1, v_0, \epsilon)$ depends continuously on (v_0, ϵ), taking $d > 0$ small enough, it holds that for all $(u; v_0, \epsilon)$ in $I \times \frac{1}{2}I \times dI$,
$$\frac{\partial v}{\partial \epsilon}(1, v_0, 0) > c \qquad \text{and} \qquad \frac{\partial v}{\partial \epsilon}(1, v_0, 0) < \tfrac{1}{2}.$$

Therefore from the Mean Value Theorem, for all $(u; v_0, \epsilon)$ in $I \times \frac{1}{2}I \times dI$

$$v(1; v_0, \epsilon) \geq v_0 + c\epsilon, \quad \text{if } \epsilon \leq 0$$
$$v(1; v_0, \epsilon) \leq v_0 + c\epsilon, \quad \text{if } \epsilon < 0 \quad \text{and} \tag{38}$$
$$|v(u; v_0, \epsilon) - v_0| \leq \frac{|\epsilon|}{2}$$

Without lost of generality, $\eta > 0$ can be assumed to be so small that for $\epsilon \in I$, $\alpha_\epsilon = \alpha + \epsilon\varphi N$ is an immersion. Notice that for all v in $v(q) + \frac{5}{6}\epsilon I$, $\rho(\frac{v - v_0}{|\epsilon|}) = 1$. Therefore for all ϵ in I and all (u, v) in $I \times (v(q) + \frac{5}{6}\epsilon I)$, $\varphi(u, v; v_0, \epsilon) = \psi(u, v; v_0, \epsilon)$. From this and $(7.15c)$ follows that the $\mathcal{M}$-mean curvature lines of $(\alpha + \psi(., v_0, \epsilon)N)|_{D'}$ and those of the $\alpha + \varphi(., v_0, \epsilon)N|_{D'}$ coincide, where $D' = v^{-1}[v(q) + \frac{5}{6}\epsilon I]$. Hence, the assertion that the map $v_\epsilon(q)$ is strictly increasing and its image contains the interval $[v(q) - 2c\epsilon, v(q) + 2c\epsilon]$ follows from equation 38.   □

Lemma 9. [**Lifting Lemma**] Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^∞ with a minimal $\mathcal{M}$-mean curvature line $\tilde{\gamma}$, oriented from a starting point q, whose $\omega-$limit set contains a nontrivial minimal recurrent $\mathcal{M}$- mean curvature line γ. Assume also that $\mathcal{M}$ is positive regular. Then given any $\eta > 0$, $p \in \gamma$ and any $\mathcal{M}$-mean curvature chart $(u, v) : (U, D, p) \to (V, I \times I, 0)$ where $I = [-1, 1]$, there is a $\mathcal{M}$-mean curvature chart $(s, t) : D' \to I \times I$ for α on $\mathbb{V}^2$ and a smooth function $\varphi : \mathbb{V}^2 \to [0, 1]$ such that:
i) the support of φ is contained in $D \cap D'$ and $||\varphi||_{2,V}$, the C^2-norm of φ on V, is less than η.
ii) There are arcs of minimal $\mathcal{M}$-mean curvature lines $[b, a] \subset [q, a] \subset \tilde{\gamma}$ such that a, b are in the arc $\{s = -1\}$ and $[b, a] \cap D' = [q, a] \cap D' \subset [a, a'] \cup [b, b']$, where a' and b' are the points on $\{u = 1\}$, defined by $t(a') = t(a)$ and $t(b') = t(b)$.

Moreover the minimal $\mathcal{M}$-mean curvature lines for α_ϵ on D' which passes through a (resp. b) meets the segment $\{u = 1\}$ at a point $v_\epsilon(a)$ (resp. $v_\epsilon(b)$) in such way that for some values of $\epsilon \in [0, 1]$, it coincides with a' and b'. See Figure 5.

Proof. See [17, 18].   □

Proposition 8. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^∞ with umbilic set $\mathcal{U}_\alpha \neq \emptyset$ and having a nontrivial minimal recurrent $\mathcal{M}$-mean curvature line γ and let A be a subset of $\mathbb{V}^2$ formed by finitely many minimal $\mathcal{M}$-mean curvature lines that are either minimal $\mathcal{M}$-mean curvature separatrix connections or minimal $\mathcal{M}$-mean curvature cycles. Assume also that $\mathcal{M}$ is positive regular. Then there is a point $p \in \gamma \setminus A$ such that given any chart $(u, v) : (U, p) \to (V, 0)$ on a neighborhood U of p, where U is disjoint of A, there is a sequence of smooth functions φ_n on $\mathbb{V}^2$, whose support is contained in U such that $||\varphi_n||_{2,V}$, the C^2-norm of φ_n, in the coordinate chart (u, v), tends to 0 and such that the immersions $\alpha_n = \alpha + \varphi_n N$ satisfy the following alternatives:*

i) α_n *has a $\mathcal{M}$-mean curvature cycle γ_n not completely contained in $\mathbb{V}^2 \setminus U$. Moreover if there is a minimal $\mathcal{M}$-mean curvature cycle of α (i.e. disjoint of U) which together with γ_n bound a cylinder in $\mathbb{V}^2$ then this cylinder contains an umbilic point of α.*
ii) α_n *has at least one minimal $\mathcal{M}$-mean curvature separatrix connection more than the immersion α does.*

Proof. See [17, 18]. ◻

Proposition 9. *Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion of class C^∞ and let A be a subset of $\mathbb{V}^2$ formed by finitely many minimal $\mathcal{M}$-mean curvature lines that are either minimal $\mathcal{M}$-mean curvature separatrix connections or minimal $\mathcal{M}$-mean curvature cycles. Assume also that $\mathcal{M}$ is positive regular.*

Then there is a sequence of immersions $\alpha_n = \alpha + \varphi_n N$, C^2-converging to α, such that the support of φ_n is disjoint from $\bar{A} = A \cup \mathcal{U}_\alpha$ and α_n has no non trivial minimal recurrent $\mathcal{M}$- mean curvature lines.

Proof. See [17, 18]. ◻

Let $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ be an immersion whose umbilic points are of type M_i, $i = 1, 2, 3$ and all tangential parabolic singularities are folded saddles, nodes and foci. A minimal $\mathcal{M}$- mean curvature separatrix Γ of an umbilic or parabolic point p is said to be *stabilized* provided:

i) it is not a minimal $\mathcal{M}$- mean curvature separatrix connection for α;
ii) its limit sets are umbilic points, parabolic point or attracting or repelling minimal $\mathcal{M}$- mean curvature cycles, and
iii) α is in the C^6-interior of the set of immersions that satisfy i) and ii); i.e., for any sequence of immersions α_n, C^6-converging to α_n, the sequence of separatrices Γ_n, of an umbilic or a tangential parabolic point p_n, converging to the separatrix Γ of p, verify i) and ii) for α_n.

Lemma 10. [Stabilization Lemma] *Any immersion $\alpha : \mathbb{V}^2 \to \mathbb{R}^3$ of class C^∞ is the C^2-limit of a sequence of immersions whose umbilic points are all of type M_i, $i = 1, 2, 3$, section 3, all tangential parabolic singularities are folded saddles, nodes and foci, and furthermore:*

i) *their minimal $\mathcal{M}$-mean curvature separatrices are all stabilized;*
ii) *the ω-limit set of any oriented minimal $\mathcal{M}$-mean curvature line is either an umbilic or a cuspidal parabolic point or a minimal $\mathcal{M}$-mean curvature cycle, and*
iii) *for any $s \geq 6$, α is in the C^s-interior of the set of immersions satisfying i) and ii).*

Proof. See [17, 18]. ◻

Remark 8. *In all lemmas and propositions above the same conclusions hold for the maximal $\mathcal{M}$-mean curvature lines provided the corresponding hypotheses are made also in this case.*

7.1. **Proof of the Density Theorem 10.**

Part 1: Elimination of nontrivial recurrences

By proposition 9 the recurrent lines can always be destroyed by a finite sequence of small local C^2-perturbations of the immersion α. Each perturbation creates either a new $\mathcal{M}$-mean curvature cycle or a new $\mathcal{M}$- mean curvature separatrix connection.

Initially will be considered the elimination of the minimal recurrent $\mathcal{M}$-mean curvature lines.

The key points involved in the argument will be given below.

Let γ be a non trivial minimal recurrent $\mathcal{M}$- mean curvature line.

Assume first that γ is *orientable*, i.e, it is possible to give an orientation in γ such that on a $\mathcal{M}$- mean curvature chart it is induced by an orientation defined locally on the $\mathcal{M}$- mean curvature line field by the chart. The recurrent lines on vector fields and those of $\mathcal{M}$- mean curvature foliations on the Torus, in section 5, are of this type. In this case there is a piecewise smooth simple closed curve of the form $[\mathbf{b}, \mathbf{a}] \cup [\mathbf{b}; \mathbf{a}]$, with $[a, b] \subset \gamma$ and $\mathbf{a}$ near $\mathbf{b}$, that can be slightly perturbed to obtain a minimal $\mathcal{M}$- mean curvature cycle for the approximating immersion. Here, and in what follows, $[\mathbf{b}; \mathbf{a}]$ means an arc of a maximal $\mathcal{M}$- mean curvature line and $[\mathbf{b}, \mathbf{a}]$ is an arc of a minimal $\mathcal{M}$- mean curvature line. The arrangement of these points are illustrated in Figure 5.a.
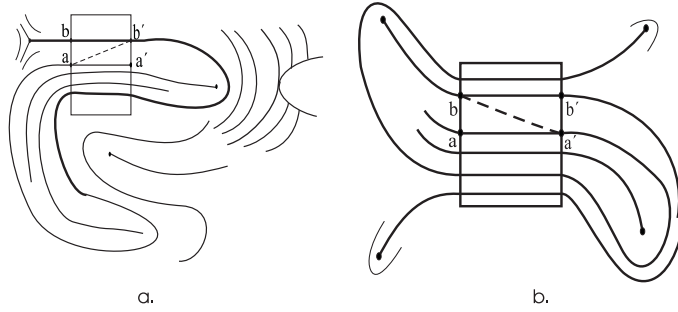

FIGURE 5.   Recurrences of $\mathcal{M}$- mean curvature lines

When the recurrence is *oscillatory* (i.e. non–orientable), then there is no such simple closed curve available. In this case there are minimal $\mathcal{M}$-mean curvature separatrices accumulating on $\mathbf{p}$. These separatrices can be connected by means of a small perturbation of the immersion. The $\mathcal{M}$- mean curvature lines, for $\mathcal{M} = \mathcal{H}$, $\mathcal{M} = \sqrt{\mathcal{K}}$ and $\mathcal{M} = \mathcal{K}/\mathcal{H}$, on the ellipsoid, presents this type of recurrence, see [11], [12] and [13]. This situation is illustrated in Figure 5. b.

The possibility of finding perturbations as those described above is established in the Lifting Lemma 9 and Proposition 8.

This is done as follows.

Consider a non trivial recurrent minimal separatrix γ' of an umbilic or parabolic point q. Take $p \in \gamma$ and a $\mathcal{M}$- mean curvature chart (u, v) : $(D, p) \to (I \times I, 0)$.

By lemma 6, arbitrarily close to $\{v = 0\}$, two points $\mathbf{a}$, $\mathbf{b}$ in $\{u = -1\} \cap \gamma'$ can be selected such that $v(\mathbf{b}) - v(\mathbf{a}) = 2r > 0$ and $(\mathbf{b}, \mathbf{a})$ has the following spacing property relatively to the maximal $\mathcal{M}$- mean curvature arc $[\mathbf{a}; \mathbf{b}]$:

$(\mathbf{b}, \mathbf{a})$ is disjoint from $\{v(\mathbf{a}) - (3/2)r < v < v(\mathbf{b}) + (3/2)r\} \cap \{u = -1\}$.

It results from this that a local version of the lifting argument (Lemma 7) can be applied to obtain, by means of an $\epsilon-$ small C^2-perturbation supported on $\{v(\mathbf{a}) - (3/2)r < v < v(\mathbf{b}) + (3/2)r\}$, the following. Given $\eta > 0$, there is a constant $c = c(\eta, (u, v)) > 0$, which does not depend on how close $\mathbf{a}$ and $\mathbf{b}$ are, such that, for every $\mathbf{x} \in [\mathbf{a}; \mathbf{b}]$, the $\mathcal{M}$- mean curvature line through the point $\mathbf{x}$ can reach any point of the segment $\{u = 1\} \cap \{v(\mathbf{x}) - 2cr < v < v(\mathbf{x}) + 2cr\}$.

Consider first the assumption that $c = 1$.

If $[\mathbf{b}, \mathbf{a}] \setminus \{v(\mathbf{a}) - \frac{3r}{2} < v < v(\mathbf{b}) + \frac{3r}{2}\} = (\mathbf{b}', \mathbf{a})$, with $\mathbf{b}' \in \{u = 1\}$, then, via a perturbation, a minimal $\mathcal{M}$- mean curvature cycle can be obtained. This is illustrated in Figure 5.a.

If, however, $[\mathbf{b}, \mathbf{a}] \setminus \{v(\mathbf{a}) - \frac{3r}{2} < v < v(\mathbf{b}) + \frac{3r}{2}\} = (\mathbf{b}', \mathbf{a}')$ with $\mathbf{b}', \mathbf{a}' \in \{u = 1\}$, it seems to be difficult to approximate $[\mathbf{b}', \mathbf{a}']$ by a minimal $\mathcal{M}$- mean curvature cycle. Nevertheless, by moving $\mathbf{a}'$ towards $\mathbf{b}'$, one can generate a continuous family of minimal $\mathcal{M}$- mean curvature arcs with endpoints in $\{u = 1\}$. In this process the resulting endpoints become close to each other but cannot coincide. Using this it is proved that the limit set of this family of minimal $\mathcal{M}$- mean curvature arcs must contain an umbilic or parabolic point $\mathbf{q}'$ and an arc $[\mathbf{p}', \mathbf{q}']$, of a minimal $\mathcal{M}$- mean curvature separatrix of $\mathbf{q}'$, intersecting

$$\{v(\mathbf{a}) - \tfrac{3r}{2} < v < v(\mathbf{b}) + \tfrac{3r}{2}\} \cap \{u = 1\}$$

at a point of $(\mathbf{b}'; \mathbf{a}')$. In this situation, via a perturbation, a minimal $\mathcal{M}$- mean curvature separatrix connection between $\mathbf{q}$ and $\mathbf{q}'$ can be produced. This is illustrated in Figure reffig:71.b.

In general, $c > 0$ is much smaller than 1 and the analysis is done by showing that lemma 7 can be used a number n of times, where n is of the order of $1/c$, to finally obtain enough lifting as to make possible the application of the arguments above. The n intervals $[\mathbf{a}_i; \mathbf{b}_i]$ which play the same role as that performed by $[\mathbf{a}; \mathbf{b}]$ and on which Lemma 7 is to be used, are described below.

If $\mathbf{a}$ and $\mathbf{b}$ are close enough to each other and $(\mathbf{a}, \mathbf{b})$ is long enough, it is proved that there is a family $\{[\mathbf{a}_t; \mathbf{b}_t]; t \in [1, n]\}$ of pairwise disjoint maximal $\mathcal{M}$- mean curvature arcs such that:

  i) $[\mathbf{a}_1; \mathbf{b}_1] = [\mathbf{a}; \mathbf{b}]$,
  ii) the curves $\mathbf{a}_t \in \tilde{\gamma}, \mathbf{b}_t \in \tilde{\gamma}$ are regular,

iii) for all $i \in \{1, 2, ..., n\}$, $[\mathbf{a}_i; \mathbf{b}_i]$ is contained in $\{u = \pm 1\}$, and $(\mathbf{b}_i, \mathbf{a}_i) \setminus$ $\{v = v(\mathbf{b}_i), v(\mathbf{a}_i)\}$ is disjoint from D_i. Here $D_i = \{v(\mathbf{a}_i) - r_i < v < v(\mathbf{b}_i) + r_i\} \subset \{-\frac{1}{2} < v < \frac{1}{2}\}$ with $2r_i = |v(\mathbf{b}_i) - v(\mathbf{a}_i)|$ and, finally,

iv) the sets D_i, $i \in \{1, 2, ..., n\}$, are pairwise disjoint.

See Figure 6 keeping in mind that D_i is the sub rectangle of D with vertical edges $\{\tilde{s} = x_i\}$ and $\{\tilde{s} = x_i'\}$. See also Figure 6 keeping in mind that D_i is the rectangle with vertices $\mathbf{z}_i'$, $\mathbf{z}_i$, $\mathbf{w}_i'$, and $\mathbf{w}_i$.

By lemma 7, the amount of lifting gained in each set D_i is $2c$ times r_i and it is carried to $[\mathbf{a}_{i+1}; \mathbf{b}_{i+1}]$, rescaled almost linearly, by the $\mathcal{M}$- mean curvature foliation. Consequently, as nc is near 1, all of these lifting can be added up as required.

So, all recurrent minimal $\mathcal{M}$- mean curvature lines can be eliminated.

To eliminate the recurrent maximal $\mathcal{M}$- mean curvature lines of $\mathbb{H}_{\alpha,2}^{\mathcal{M}}$, it is necessary to perform the same deformation analysis as above, applied to this case, with no fundamental change.

The $\mathcal{M}$- mean curvature separatrices of $\mathbb{H}_{\alpha,1}^{\mathcal{M}}$ are stabilized taking care to consider the C^2-deformations of the immersion α, with support in $\mathcal{M}$- mean curvature charts disjoint from the nowhere dense set A (see proposition 9) consisting of the minimal $\mathcal{M}$- mean curvature separatrices and the minimal $\mathcal{M}$- mean curvature cycles. So the stabilized minimal $\mathcal{M}$- mean curvature separatrices and minimal $\mathcal{M}$- mean curvature cycles are preserved and these deformations do not produce any new non trivial recurrence for the minimal $\mathcal{M}$- mean curvature lines.

Therefore the immersion α can be approximated in the $C^2 - topology$ by an immersion α_1 having all minimal and maximal $\mathcal{M}$- mean curvature separatrices stabilized.
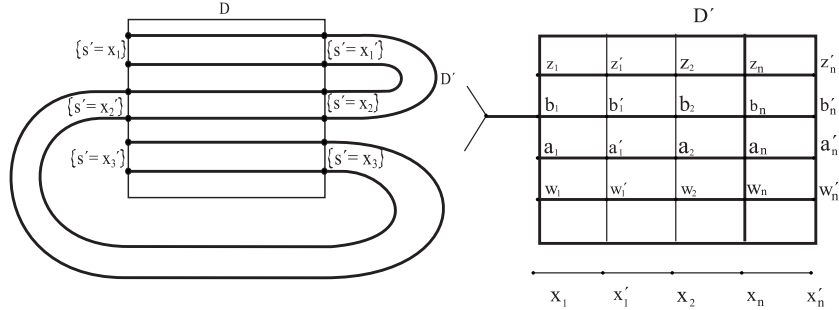

FIGURE 6.   Lifting of $\mathcal{M}$- mean curvature lines

Part 2: Conclusion of the Proof of Theorem 10

The first step is to approximate an immersion α of compact and oriented surface $\mathbb{V}^2$ by an immersion having all umbilic points of the type M_i, $i = 1, 2, 3$, proposition 1, and all parabolic points of types as described in propositions 4, 5, 6 and 7.

This can be done by the Transversality Theorem establishing the condition $T = b(b - a) \neq 0$ and by a finite number of small local changes on the coefficients of the third jet of the immersion at the umbilic points. Similar for parabolic points, see [3].

Next approximate the immersion α in C^s-sense by an analytic immersion, which will be denoted by α_1. There are two cases to consider.

Case 1. The region $\mathbb{EV}_{\alpha_1}^2$ is diffeomorphic to an annulus and the immersion α_1 is without umbilics and tangential parabolic singularities.

By using proposition 9 it is possible to obtain an analytic immersion α', C^2 close to α_1 having only finitely many $\mathcal{M}$- mean curvature cycles, all of which have finite multiplicity.

The resulting immersion α' can be deformed around a $\mathcal{M}$- mean curvature cycle to a obtain an immersion with a hyperbolic $\mathcal{M}$- mean curvature cycle. If this immersion is approximated by an analytic one, α'', will have only finitely many $\mathcal{M}$- mean curvature cycles, all of which with finite multiplicity. In either case, using proposition 3, α'' can be approximated by an immersion $\tilde{\alpha}$, all whose $\mathcal{M}$- mean curvature cycles are hyperbolic, which belong to the class $\mathcal{G}_i(\mathbb{V}^2)$, $i = 1, 2$, since conditions i), iii), iv) and v) are guaranteed by the Stabilization Lemma 9. This ends the proof in this case.

Case 2. The analytic immersion α_1 has separatrices which can be associated to umbilic points, all are of the types M_i, $i = 1, 2, 3$, or parabolic tangential singularities as in section 5.

In this case, as shown in Part 1, the immersion α_1 can be taken so that both, minimal and maximal $\mathcal{M}$- mean curvature separatrices are stabilized and without non trivial recurrences. The next step, using proposition 3, is to deform the immersion in order to obtain an immersion with all minimal and maximal $\mathcal{M}$- mean curvature cycles hyperbolic. This ends the proof.

Remark 9. *With the proof of Theorem 10 for a fairly general $\mathcal{M}$, we have completed the density results formulated in [12] and [13] for the particular cases $\mathcal{M} = \sqrt{\mathcal{K}}$ and $\mathcal{M} = \frac{\mathcal{K}}{\mathcal{H}}$.*

8. Concluding Remarks

This paper presents a theory of unification and generalization for the *classical mean curvature* lines on a surface immersed in $\mathbb{R}^3$. See section 1 and the papers [10, 11, 12, 13, 14]. To this end the notion of *mean curvature function* was introduced, assimilating and adapting for our purposes in Geometry the general properties for the *Means* studied in Arithmetic and Analysis. See Chapter 8 of Borwein and Borwein [4].

The Structural Stability, with its well established achievements in the Differential Equations of Geometry, has been set as a primary goal and a test for the penetration of the generalization proposed in this paper. The methods developed in previous specific works dealing with the *classical means* have been further elaborated and adapted to apply to the general case of differential equations of $\mathcal{M}$-*mean curvature lines* treated here.

It has been established that the *umbilic points* and the *cycles* of the $\mathcal{M}$-mean curvature foliations present a remarkable analogy with those of the classical *Arithmetic, Geometric* and *Harmonic - mean curvature* corresponding cases. See sections 3 and 4.

The study of the parabolic singularities revealed new interesting aspects. The analysis presented here was possible by imposing special regularity conditions on the general *mean curvature function* $\mathcal{M}$. Nevertheless, significant cases were covered. The results achieved, however, do not apply to the classic *AG mean* (example 2), which has a parabolic pattern not reducible to algebraic form called *1/k-regularity*. See section 5 for the study of the cases $k = 1, 2$.

The development of the transcendental analysis needed to study the parabolic singularities of *AG mean* can be regarded as the first problem of interest left open in this paper.

A great deal of problems – such as the studies of bifurcations and of immersions of higher dimension and co-dimension – already proposed in the particular cases of *classical mean curvatures*, make sense and have a renewed challenge in the present generalized setting.

By taking the function $\mathcal{M}$ as a functional parameter, or itself depending or a real parmeter, as in the case of the *Holder Mean of order r* denoted $\mathcal{H}_r$ in example 1, the bifurcation analysis of the transition between different classical differential equations of geometry and pertinent foliations with singularities gains new vitality.

The most intriguing of these problems is the *Closing Lemma*. In fact, to prove how to raise the class proximity class from C^2 to C^3 in Theorem 10 is not known even for the case principal foliations. See section 7 and [17, 18]. Also to achieve the C^1 density for Structural Stability of folded, "billiard", non-convex configurations is also an open problem in all cases of nets, including *asymptotic* ones. See section 6 and [14].

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